Theory and Applications of Decoupling Fields for Forward-Backward Stochastic Differential Equations

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Abstract

This thesis deals with the theory of so called forward-backward stochastic differential equations (FBSDE) which can be seen as a stochastic formulation and in some sense generalization of parabolic quasi-linear partial differential equations. The thesis consist of two parts: In the first we develop the theory of so called decoupling fields for general multidimensional fully coupled FBSDE in a Brownian setting. The theory consists of uniqueness and existence results for decoupling fields on the so called the maximal interval. It also provides tools to investigate well-posedness and regularity for particular problems.

In total the theory is developed for three different classes of FBSDE: In the first Lipschitz continuity of the parameter functions is required, which at the same time are allowed to be random. The other two classes we investigate are based on the theory developed for the first one. In both of them all parameter functions have to be deterministic. A form of local Lipschitz continuity replaces the more restrictive Lipschitz continuity for the second class, while boundedness of the terminal condition is required. This assumption is dropped in the third class, but the assumption of local Lipschitz continuity is partially restricted as a consequence.

In the second part we apply these techniques to three different problems: In the first application we demonstrate how well-posedness of FBSDE in the so called non-degenerate case can be investigated. As a second application we demonstrate the solvability of a system, which after an additional investigation of smoothness properties of the associated decoupling field provides a solution to the so called Skorokhod embedding problem (SEP) via FBSDE. The solution to the SEP is provided for the case of general non-linear drift. The third application provides solutions to a complex FBSDE from which optimal trading strategies for a problem of utility maximization in incomplete markets are constructed. The FBSDE is solved in a relatively general setting, i.e. for a relatively general class of utility functions on the real line.

In these applications we heavily and frequently use the aforementioned theoretical base.

Zusammenfassung

Diese Arbeit beschäftigt sich mit der Theorie der sogenannten stochastischen Vorwärts-Rückwärts-Differentialgleichungen (FBSDE), welche als ein stochastisches Anologon und in gewisser Weise als eine Verallgemeinerung von parabolischen quasi-linearen partiellen Differentialgleichungen betrachtet werden können. Die Dissertation besteht aus zwei Teilen: In dem ersten entwicklen wir die Theorie der sogenannten Entkopplungsfelder für allgemeine mehrdimensionale stark gekoppelte FBSDE. Diese Theorie besteht aus Existenz- sowie Eindeutigkeitsresultaten für Entkopplungsfelder basierend auf dem Konzept des maximalen Intervalls. Es beinhaltet darüberhinaus Werkzeuge um Singulartätsfreiheit und Regularität von konkreten Problemen zu untersuchen.

Insgesamt wird die Theorie für drei verschiedene Klassen von Problemen entwickelt: In dem ersten Fall werden Lipschitz-Bedingungen an die Parameter des Problems vorausgesetzt, welche zugleich vom Zufall abhängen dürfen. Die Untersuchung der beiden anderen Klassen basiert auf dem ersten. In diesen werden die Parameter als deterministisch vorausgesetzt. Gleichwohl wird die Lipschitz-Stetigkeit durch lokale Lipschitz-Stetigkeit in dem zweiten der drei Fälle abgeschwächt und Beschränktheit der Endbedingung vorausgesetzt. Diese letzte Voraussetzung kann in dem letzten Fall aufgehoben werden, jedoch ist die lokale Lipschitz-Stetigkeit dann nur zum Teil gültig.

In dem zweiten Teil werden diese abstrakten Resultate auf im Wesentlichen drei konkrete Probleme angewendet: In der ersten Anwendung wird gezeigt wie globale Lösbarkeit von FBSDE in dem sogenannten nicht-degenerierten Fall untersucht werden kann. In der zweiten Anwendung wird die Lösbarkeit eines gekoppelten Systems gezeigt, welches nach einer zusätzlichen Untersuchung der Glattheit des zugehörigen Entkopplungsfeldes eine Lösung zu dem Skorokhod'schen Einbettungproblem liefert. Die Lösung wird für den Fall einer allgemeinen nicht-linearen Drift konstruiert. Die dritte Anwendung führt auf Lösbarkeit eines komplexen gekoppelten Vorwärt-Rückwärts-Systems, aus welchem optimale Strategien für das Problem der Nutzenmaximierung in unvollständingen Märkten konstruiert werden. Das System wird in einem verhältnismäßig allgmeinen Rahmen gelöst, d.h. für eine verhältnismäßig allgemeine Klasse von Nutzenfunktion auf den reellen Zahlen.

In all diesen Anwendungen wird starker und häufiger Gebrauch der zuvor entwickelten Theorie gemacht.

Contents

Chapter 1

Introduction

In recent decades Backward Stochastic Differential Equations (BSDE) and more generally Forward-Backward Stochastic Differential Equations (FBSDE) have been studied extensively. They have many applications in various fields of applied mathematics, such as stochastic control theory and mathematical finance. Moreover, they are closely connected to an important class of partial differential equations. This thesis is to a great extent concerned with the theoretical analysis of FBSDE and their solutions. Before we go into more detail, let us briefly discuss two seemingly unrelated problems, which lead to this type of equations:

In financial mathematics one often starts with an optimization problem, which involves randomness and reduces it to an FBSDE. For instance, one common problem is the one of maximizing expected utility from trading in financial markets. Mathematically this can be expressed as

$$\sup_{\pi\in\mathcal{A}} \mathbb{E}[U(X_T^\pi + H)],$$

where $U:\mathbb{R}\to\mathbb{R}$ is some utility function, H the (random) initial endowment and X_T^π is the (random) wealth given by the initial wealth $x\in\mathbb{R}$ and the trading strategy $\pi\in\mathcal{A}$, where $\mathcal{A}$ is the set of admissible strategies. So, the problem is about finding an optimal trading strategy π^*. In general, one assumes that the market is incomplete in the sense that H cannot be perfectly replicated by trading in the market, due to external sources of randomness which are not accessible by the investor. Under certain assumptions on $U, H, \mathcal{A}$ one can show that an optimal strategy π^* satisfies the rather complex system of equations

$$\begin{aligned}
X_t &= x - \int_0^t \left(\pi_1(\theta_s)\frac{U'(X_s+Y_s)}{U''(X_s+Y_s)} + \pi_1(Z_s)\right)^\top \mathrm{d}W_s - \int_0^t \left(\pi_1(\theta_s)\frac{U'(X_s+Y_s)}{U''(X_s+Y_s)} + \pi_1(Z_s)\right)^\top \pi_1(\theta_s)\,\mathrm{d}s,\\
Y_t &= H - \int_t^T Z_s^\top\,\mathrm{d}W_s - \int_t^T \Bigg[-\frac{1}{2}|\pi_1(\theta_s)|^2\frac{U^{(3)}(X_s+Y_s)|U'(X_s+Y_s)|^2}{(U''(X_s+Y_s))^3}\\
&\qquad + |\pi_1(\theta_s)|^2\frac{U'(X_s+Y_s)}{U''(X_s+Y_s)} + Z_s\cdot\pi_1(\theta_s) - \frac{1}{2}|\pi_2(Z_s)|^2\frac{U^{(3)}}{U''}(X_s+Y_s)\Bigg]\,\mathrm{d}s, \qquad (1.1)
\end{aligned}$$

where π^* is closely connected to the solution component Z: In order to obtain an optimal strategy π^* one has to solve the above stochastic differential equation by constructing solution processes X, Y, Z and then set $\pi^*_s := -\pi_1(\theta_s)\frac{U'}{U''}(X_s+Y_s) - \pi_1(Z_s)$ to obtain the optimal strategy.

Consult [HHI$^+$14] for more details.

Let us next consider the following fundamental problem in probability theory: Given a Brownian motion W and a probability measure ν on $\mathbb{R}$ find a stopping time τ such that W_τ has the distribution ν. This problem is referred to as the Skorokhod embedding problem. Different constructions have been proposed in this context. In general, one is interested in making τ as "small" as possible, i.e. make it satisfy certain minimality criteria. One popular approach, initially proposed in [Bas83] (see also [AHI08] and [AHS13]), is to construct processes Y, Z such that

$$Y_t = g(W_1) - \int_t^1 Z_s \,\mathrm{d}W_s, \quad t \in [0,1], \tag{1.2}$$

where g is chosen in such a way that $g(W_1)$ has distribution ν. Since the random time-change in the quadratic variation scale $\int_0^\cdot Z_s^2 \,\mathrm{d}s$ transforms the martingale $\int_0^\cdot Z_s \,\mathrm{d}W_s$ into a Brownian motion B we can set $\tilde{\tau} := \int_0^1 Z_s^2 \,\mathrm{d}s$ and obtain $B_{\tilde{\tau}} = g(W_1)$ such that $B_{\tilde{\tau}}$ has the prescribed law.
However, if we want to solve the problem in a more general setting, for instance by assuming that the process to be stopped is a Gaussian process G of the form $G_t = G_0 + \int_0^t \alpha_s \,\mathrm{d}s + \int_0^t \beta_s \,\mathrm{d}W_s$ with deterministic α, β, the associated dynamical system becomes more complex. More precisely, we will have to find processes X, Y, Z such that X is 2-dimensional and

$$X_s^{(1)} = \int_t^s 1 \,\mathrm{d}W_r, \qquad X_s^{(2)} = \int_t^s Z_r^2 \,\mathrm{d}r, \qquad Y_s = g(X_1^{(1)}) - \delta(X_T^{(2)}) - \int_s^1 Z_r \,\mathrm{d}W_r, \tag{1.3}$$

where the functions $g, \delta : \mathbb{R} \to \mathbb{R}$ are given by ν, α, β.

Systems like (1.1) and (1.3) represent so called Forward-Backward Stochastic Differential Equations (FBSDE). A general FBSDE in a Brownian setting is a system of the form

$$X_t = x + \int_0^t \mu(s, X_s, Y_s, Z_s) \,\mathrm{d}s + \int_0^t \sigma(s, X_s, Y_s, Z_s) \,\mathrm{d}W_s,$$

$$Y_t = \xi(X_T) - \int_t^T f(s, X_s, Y_s, Z_s) \,\mathrm{d}s - \int_t^T Z_s \,\mathrm{d}W_s,$$

where X and Y can be multidimensional, such that the two equations above represent systems of equations in general. The Brownian motion W can also be multidimensional. The first equation is referred to as the forward equation and the second as the backward equation. The nature of the underlying problem is encoded in the parameter functions μ, σ, f, the initial condition $x \in \mathbb{R}^n$ and the terminal condition ξ.

The system is called decoupled if either μ, σ do not depend on Y, Z, or if ξ, f do not depend on X. In these two cases the problem can be treated by solving one of the equations first, and then simply plugging the solution processes obtained into the other equation in order to solve the latter in the second step.

Let us remark that the theory of decoupled problems is much more extensive than the theory of general strongly coupled FBSDE, in which still many questions remain unanswered. Consult, for instance, [EPQ97] for classical results on BSDEs. This thesis, however, is primarily concerned with developing and applying techniques to treat general strongly coupled systems. The reason is the following: While many of the particular problems, e.g. optimization problems, in applied mathematics lead to decoupled FBSDEs (e.g. BSDEs) under special structural assumptions, the general case usually exhibits some form of coupling. For instance, (1.1) can be reduced to a decoupled problem if we assume that U is an exponential utility function (see [HIM05]). Unfortunately many of the standard results and techniques used to study decoupled problems cannot be straightforwardly extended to general systems and remain specific for this class of problems. Clearly, another level of complexity is required to treat the general case.

What makes the treatment of general systems so complicated, however, is the fact that coupled FBSDE do not always have solutions. Their solvability cannot be ensured by merely making analytic assumptions on the parameters of the problem (e.g. Lipschitz continuity). Instead, structural assumptions have to be made as well. For instance, one can construct solutions if T is sufficiently small.

1.1 Method of decoupling fields

In Chapter 2 we propose a method to analyze the solvability of coupled systems. Similarly to [MWZZ11] we work with a decoupling field u, which roughly speaking, is a function, which connects Y and X via the decoupling condition $Y_s = u(s, X_s)$. So, u is a function of $\omega \in \Omega$, time and space. Accordingly we introduce a different notion of solvability: Instead of merely requiring the existence of processes X, Y, Z satisfying the FBSDE, we also seek to obtain a function u as above. This additional structure exhibits properties not present in classical solutions. It allows to employ the following recipe to construct solutions to FBSDE: Instead of solving the problem on the whole interval $[0, T]$ directly, divide $[0, T]$ into finitely many sufficiently small intervals. Then construct decoupling fields on every subinterval subsequently starting at the right boundary of $[0, T]$ and then moving to the left. Finally one can patch these decoupling fields together and obtain a decoupling field on the large interval. The key property of decoupling fields used in this argument is that they are closed under the operation of concatenation.

In other words the main advantage of using decoupling fields is that the processes X, Y, Z do not have to exist globally but only locally, i.e. on small intervals, for the decoupling field to make sense. On the other hand if they exist locally, they also exist globally on the compact interval the decoupling field is defined on, as one can show. As we concatenate decoupling fields on small intervals from right to left the underlying processes X, Y, Z are also constructed on larger and larger intervals automatically along the way. Heuristically, this is related to several works in the literature (e.g. [Del02], [MWZZ11]). The goal of Chapter 2 is to provide a sufficiently thorough theory to be employed as the underlying theoretical machinery to treat the complex problems we have touched upon in the introduction. We will discuss later, why we could not use existing techniques for this purpose.

Now, the actual method we propose in Chapter 2 consists of the following four steps:

1. Assume indirectly that a given system does not have regular solutions on $[0, T]$. This implies that the problem can be solved on some non-empty interval of the form $(t_{\min}, T]$. Choose arbitrary $t \in (t_{\min}, T]$, $x \in \mathbb{R}^n$ and consider the corresponding FBSDE.

In other words the FBSDE can still be solved properly on every interval $[t, T]$ as long as $t \in (t_{\min}, T]$. Since we assume that this cannot be extended to $[t_{\min}, T]$ a *singularity* must occur at $t_{\min}$, which can be expressed in terms of the behaviour of the spatial derivative of the decoupling field at the left boundary.

2. Differentiate the FBSDE w.r.t. the initial value x. We obtain joint dynamics of $\frac{\mathrm{d}}{\mathrm{d}x}X$, $\frac{\mathrm{d}}{\mathrm{d}x}Y$, $\frac{\mathrm{d}}{\mathrm{d}x}Z$.

3. Using Itô's formula deduce the dynamics of $\frac{\mathrm{d}}{\mathrm{d}x}Y_s(\frac{\mathrm{d}}{\mathrm{d}x}X_s)^{-1}$. This process can be expected to coincide with $u_x(s, X_s)$ as a consequence of the chain rule applied to the decoupling condition $Y_s = u(s, X_s)$.

4. Using the dynamics of $u_x(s, X_s)$, show that its modulus can be bounded from above in some sense independently of t, x, s, ω. This rules out a singularity and creates a contradiction to our assumption.

However, in order to execute this recipe, we obviously need to develop a theoretical basis and clarify several points:

Firstly, decoupling fields together with the associated X, Y, Z have to be constructed on the so called maximal interval. It is either $[0, T]$ or has the form $(t_{\min}, T]$ as above. This technique is a natural consequence of the fact that, as is well-known, solutions might not exist on the whole interval $[0, T]$ and that, even if the problem is well-posed, the reasons for this can be quite delicate as we will see in subsequent chapters. Therefore, according to our paradigm well-posedness is a matter of an a

posteriori study. In other words: First we apply the theory which works regardless of whether the particular problem is well-posed or not and only then we investigate potential well-posedness.
Also:

- Regularity of u, X, Y, Z should be investigated. In particular, we need differentiability w.r.t. the initial value x for the above method to work. Since we work under Lipschitz or merely local Lipschitz conditions and also do not assume non-degeneracy of σ, we cannot expect classical differentiability. Therefore, the use of *weak derivatives* appears as an alternative.

- Behaviour of the spatial derivative u_x at the left boundary of the maximal interval has to be analyzed.

These points will be clarified in Chapter 2.

The above method is applicable to so called SLC[1] problems, i.e. problems which satisfy certain standard Lipschitzian assumptions. However, many practical problems exhibit non-linearities in μ, σ, f in the sense that they might be only locally Lipschitz continuous in Y, Z. We can expand our theory to these so called SLLC problems under the assumption of μ, σ, f being deterministic, i.e. assuming that they do not depend on $\omega \in \Omega$. We call such problems Markovian. Due to non-linearities in Y in the SLLC case a singularity might occur not only in u_x but also in u itself. Therefore, there is an additional step for SLLC problems:

0. Using the dynamics of Y, deduce a uniform bound for u.

Sometimes this step has to be performed before step 4., for instance, at the very beginning, because controlling u_x might rely on controlling u (cf. Chapter 3), but in other cases this step 0. can also be performed at the end.

Furthermore, we will discuss so called MLLC problems in Chapter 4, which are Markovian problems as well, but with the property of being only locally Lipschitz continuous in Z and in some sense Lipschitz continuous in X, Y. The only advantage of this "MLLC theory" is that, unlike in the case of SLLC, the boundedness of ξ does not have to be assumed. Accordingly, step 0. can be skipped. Apart from that, MLLC theory can be considered inferior to SLLC for the following reason: We have observed that regularity assumptions on ξ are a smaller problem than on μ, f, σ. For instance, boundedness of ξ for the Skorokhod embedding problem can be achieved by assuming that the target measure ν has compact support on $\mathbb{R}$, which is a reasonable approximative assumption assuming that this distribution is light-tailed, like the normal distribution for example. In the case of utility maximization, the boundedness of ξ translates into the boundedness of the initial endowment, which is often assumed anyway, for instance due to the quadratic dependence on Z in f. The structure of μ, f, σ, however, is mainly determined by the nature of the underlying problem and is hard to manipulate: In Chapter 5 we have artificially imposed Lipschitz continuity on f to obtain SLC which resulted in significant additional work to show that the trading strategies obtained from this truncated problem are still optimal in some sense. In other words, having a theory which allows for a more general dependence structure in Y, Z for μ, σ, f is probably more useful than a theory which allows more general ξ.

In this thesis we use the method of decoupling fields for purposes that go beyond showing existence of solutions to particular FBSDEs: For instance, as we will see in Chapter 4, which deals with the Skorokhod embedding problem, existence of X, Y, Z only provides weak solutions to this problem. In order to have strong solutions, more structure is required. This is achieved by investigating certain regularity properties of the associated decoupling field u. In [Bas83] and [AHI08] this is done via the PDE satisfied by u. This works because the problem can be reduced to a non-degenerate one due to the special structure of the drift.

[1] SLC stands for Standard Lipschitz Conditions

1.2 The structure of this thesis

In Chapter 2 we develop the underlying theory of decoupling fields later utilized in the remaining chapters. The theory is largely self-contained. We begin with a local existence and uniqueness result (Theorem 2.2.1) constructing decoupling fields on small intervals and capturing important properties, which come as a by-product of the construction. After expanding these local results to global analogues, we develop the SLLC theory for Markovian problems. Let us also remark that some of the arguments in the proof of local existence and uniqueness would not have been applicable if were to work with Markovian problems right from the start.

As already indicated, we will work extensively with the notion of weak differentiability in Chapter 2. We will comment on the reasons for this in the introductory remarks to that chapter. Since working with weak derivatives w.r.t the initial value in our probabilistic setting probably cannot be considered standard, we show several rules for weak derivatives in the appendix.
The main results of Chapter 2 are Theorem 2.5.11 in the SLC case and Theorem 2.5.29 in the Markovian case (SLLC version). They provide existence, uniqueness and regularity on the maximal interval. These results are supplemented by Lemmas 2.5.12 and 2.5.30, which provide criteria to investigate well-posedness.

The remaining three chapters contain applications of the theory developed in Chapter 2. In every chapter we include a compact summary of the relevant theory regarding decoupling fields. This creates some redundancy but brings about the benefit that the Chapters 2 to 5 can be read in any order. It is advisable to read Chapters 5 and 4 before moving to the more theoretical Chapter 2, which might appear unnecessarily detailed otherwise.

In Chapter 3 we look at a coupled but at the same time non-degenerate problem and demonstrate how global existence can be shown using non-degeneracy without relying on PDE theory. This provides a stochastic interpretation of the regularizing properties of a non-degenerate σ. The main result is Theorem 3.2.1.

Chapters 4 and 5 provide solutions to the problems (1.1) and (1.3) via decoupling fields. Both chapters contain an introduction to the corresponding problems they originate from, so we refer to the beginning of these chapters for more details on this matter. Let us nevertheless make a few comments on the actual treatment of the particular FBSDEs in Chapters 4 and 5:

In Chapter 4, which is based on a joint work with D.J. Prömel and P. Imkeller, we allow ourselves to develop some additional theory and discuss the aforementioned MLLC case. MLLC stands for Modified Local Lipschitz Conditions. We, thereby, distinguish them from problems satisfying SLLC (i.e. Standard Local Lipschitz Conditions), for which local Lipschitz continuity in both Y and Z is allowed. Both SLLC and MLLC require the problem to be Markovian. However, the use of MLLC makes it possible to avoid assuming boundedness of ξ in Chapter 4. At the same time, it unfortunately requires the use of a cutoff while studying certain MLLC problems appearing in our analysis of higher order derivatives of u. This creates some additional work which could have been avoided by assuming more restrictive SLLC in the beginning.
Based on our analysis of these systems, we construct solutions to the Skorokhod embedding problem via FBSDE for the case of general non-linear drift. The solvability of the particular FBSDE is provided by Lemma 4.3.1. According to its proof the key feature of the coupled system, which creates well-posedness is that the process $u_x(s, X_s)$ is a martingale w.r.t. an equivalent probability measure. However, Lemma 4.3.1 merely leads to weak solutions of the Skorokhod embedding problem. In order to solve the problem in its strong sense we demonstrate solvability of two more coupled systems, which are in a way associated with the spatial derivatives of the decoupling field. The main results are Theorem 4.3.6 and Lemma 4.3.10 according to which we obtain a strong solution if the parameters of the embedding problem are sufficiently regular.

In Chapter 5, which is based on a joint work with P. Imkeller, we investigate FBSDEs appearing

in the problem of utility maximization under two different sets of conditions and employ SLC theory in the first and MLLC in the second. However, we could have worked with SLLC in the latter case as well, since we prove uniform boundedness of Y anyway. The FBSDE is solved for a class of utility functions described in Remark 5.2.4. The main result is Theorem 5.3.13, which solves the problem in the second, i.e. the Markovian, case. In this situation we assume that the parameters of the problem depend on the random scenario ω only through a standard, possibly high-dimensional, diffusion.
Heuristically the argumentation for the Markovian case is similar to the first situation, where the FBSDE is truncated to ensure Lipschitz continuity. However, due to high-dimensionality of the joint system, the calculations become more complex. Also, super-linear growth in Z requires a deeper exploitation of the particular structure of the system in order to show well-posedness. The benefit, however, is that the trading strategies obtained from solving this FBSDE are optimal under the initial measure $\mathbb{P}$, rather than some probability measure close to it (see Lemma 5.2.3). Furthermore, we obtain uniform boundedness of the trading strategy as a direct consequence of the boundedness of the control process Z in the Markovian case.
For both of the two cases in Chapter 5 we prove an abstract result first (Theorem 5.2.1 resp. Theorem 5.3.12) to highlight the structural properties of the corresponding FBSDE which create well-posedness. These abstract results are then applied to the actual problems (Theorems 5.2.2 and 5.3.13).

In the appendix we collect some known facts about BMO - processes and prove some auxiliary results involving these objects. We also formulate and prove results about weak derivatives. In particular, this involves chain rules and rules regarding interchanging differentiation with other operators, e.g. stochastic integrals. Moreover, the appendix contains a result about convergence of probability measures, which is needed for Chapter 5.

1.3 Existing techniques

Let us now recall some of the techniques proposed to study general coupled FBSDE.

The so called *Four Step Scheme* (e.g. [MPY94]) is based on reducing the FBSDE to a quasi-linear parabolic PDE associated with the problem. This works for parameter functions which are deterministic and sufficiently smooth. Furthermore, non-degeneracy of σ is required to obtain solutions to the PDE. This method relies heavily on [LSU68]. The work [Del02] follows a similar line and requires also that σ does not depend on Z. We will discuss the connection to PDEs in Section 1.4 in more detail. It is natural to ask why in this thesis we avoid relying on these techniques, which are based on shifting the theoretical burden to PDE theory. The problem is that certain structural requirements are made: In particular, σ is required to be *non-degenerate*, i.e. the operator norm of σ must be bounded away from 0 in some sense. Moreover, in some works σ does not depend on Z.
This is problematic in light of (1.1) and (1.3): For both problems σ is (possibly) degenerate and, furthermore, depends on Z in (1.1).
Note that the non-degeneracy of σ has a smoothening effect on the associated PDE. This means that the function solving the PDE is sufficiently smooth for the PDE to make sense. It turns out that for degenerate problems some function (i.e. the decoupling field) which, if sufficiently smooth, would satisfy the PDE still exists, even though the PDE might not make sense in classical or weak sense due to a lack of smoothness. This impedes the use of classical PDE theory in our context (even for Markovian problems).

Let us also remark that in [Del02] a technique of patching together solutions on small intervals is employed. Also, locally Lipschitz problems under certain conditions are considered.

In contrast to these techniques, the *Method of Continuation* (e.g. [HP95] or Chapter 6 in [MY99]) is purely probabilistic, but goes along with monotonicity assumptions for the parameter functions that might be hard to verify in practice. For instance, neither of the two problems (1.1) and (1.3) will satisfy these monotonicity conditions in general.

A method introduced in [Zha06] is also purely stochastic, even though it is reminiscent of techniques employed in [Del02]. Again, structural assumptions are made which will not be satisfied by the problems of our interest.

Some of the aforementioned techniques employ the so-called *Contraction Method*, which is based on constructing processes X, Y, Z via a contractive Picard-iteration that provides these processes in the limit (see [Ant93], [PT99]). However, this works only if T is sufficiently small.

In order to develop a general technique, Zhang et al. have introduced the concept of decoupling fields in [MWZZ11]. They are used to extend the *Contraction Method* proposed by Antonelli [Ant93] to construct solutions on large intervals by patching together solutions given on small intervals similar to [Zha06] and [Del02]. In [MWZZ11] the emphasis is on well-posedness: the authors are primarily interested in problems which have solutions on the whole interval $[0, T]$ and propose sufficient conditions based on the so called *characteristic BSDE* and the *dominating ODE*. Furthermore, they concentrate on one-dimensional problems. Since their construction is reminiscent of the one we will conduct in Chapter 2, let us make a few remarks on this work and how it relates to Chapter 2 of this thesis:

As mentioned, in [MWZZ11] the notion of a decoupling field, i.e. of a possibly random mapping u which among others satisfies $Y_s = u(s, X_s)$, is introduced. In order to understand under what conditions u becomes Lipschitz continuous in the second component one analyzes the dynamics of the so called *characteristic BSDE* heuristically. This BSDE essentially reflects the dynamics of the process $\frac{u(s,X_s)-u(s,\tilde{X}_s)}{X_s-\tilde{X}_s}$, where $\tilde{X}$ is associated with some initial value $\tilde{x} \neq x$. Later on, a construction of u and the associated processes X, Y, Z is discussed for small T. The proof relies on [Ant93], where rather special μ, σ, f, ξ have been considered (from our Brownian point of view). Assuming that the characteristic BSDE can be controlled (for which different sufficient conditions are provided) solutions to the FBSDE are constructed by patching together decoupling fields on small intervals.

Note the difference between the use of difference quotients of the type $\frac{u(s,X_s)-u(s,\tilde{X}_s)}{X_s-\tilde{X}_s}$ and weak derivatives $\frac{\mathrm{d}}{\mathrm{d}x}Y_s$, $\frac{\mathrm{d}}{\mathrm{d}x}X_s$ w.r.t. the initial value, which we will introduce in Chapter 2. Both fulfill a similar function. The first technique has the obvious advantage of avoiding elaborate constructions and argumentations to show weak differentiability of X, Y, Z as will have to be done in Chapter 2. Yet, we will not work with difference quotients in this thesis, for the following reasons: Firstly, it is not clear how and if this can be appropriately generalized to multidimensional problems with vector-valued processes. Moreover, in Chapters 4 and 5 we have found it convenient to work with actual derivatives, which turn out to satisfy various quite subtle relationships among each other. It is uncertain that this can be translated into relationships between difference quotients.

We would also like to remark that in this thesis we work under a slightly different definition of the notion of a decoupling field compared to the one used in [MWZZ11] for technical reasons. We hope this does not cause confusion.

1.4 Decoupling fields and PDEs

In light of the current literature it is natural to ask how FBSDEs and decoupling fields relate to PDEs. However, this work does not directly deal with the theory of partial differential equations (PDE) nor does it rely on it. So we will merely make a few heuristic remarks on this matter:

Decoupling fields to FBSDEs are in some sense related to quasi-linear partial differential equations of parabolic type:

$$u_t(t,x) + \sum_{i=1}^{n} \tilde{\mu}_i(t,x,u(t,x),u_x(t,x))u_{x_i}(t,x) + \sum_{i,j=1}^{n} \tilde{\sigma}_{ij}(t,x,u(t,x),u_x(t,x))u_{x_i,x_j}(t,x) =$$
$$= \tilde{f}(t,x,u(t,x),u_x(t,x)), \qquad u(T,\cdot) = \tilde{\xi}$$

where $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}^m$ is the solution to be found, while the parameters $\tilde{\mu}, \tilde{\sigma}, \tilde{f}, \tilde{\xi}$ are connected to μ, σ, f, ξ, such that $\tilde{\sigma}$ is symmetric non-negative definite and $\tilde{\xi}$ provides the boundary condition. Due to the wide range of applications in applied sciences an extensive theory of such problems has been developed long before the introduction of FBSDE. It is a common approach to rely on PDE theory while studying FBSDE (e.g. consult [MZ11], [MPY94], [Del02]). For instance, [MPY94] and [Del02] rely on the extensive analysis of the above system from [LSU68], where solutions to the parabolic PDE are constructed using the theorem of Leray-Schauder and rather technical estimates incorporating non-degeneracy of $\tilde{\sigma}$. This solution u obtained in a purely deterministic setting is then used to construct the associated stochastic processes X, Y, Z.

However, let us for the following look at PDEs from the perspective of FBSDEs: As an example assume that σ vanishes while f, μ do not depend on ω, Y, Z but only on s, X_s. Then the forward-backward system attains the form

$$X_t = x + \int_0^t \mu(s, X_s)\,\mathrm{d}s, \quad Y_t = \xi(X_T) - \int_t^T f(s, X_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s, \quad t \in [0,T].$$

The process X clearly satisfies a deterministic ordinary differential equation (ODE) written in the integral form. In particular, $\xi(X_T) - \int_t^T f(s, X_s)\,\mathrm{d}s$ is a deterministic value, which implies $Z = 0$. Using the decoupling condition $Y_s = u(s, X_s)$ we end up with

$$X_t = x + \int_0^t \mu(s, X_s)\,\mathrm{d}s, \quad u(t, X_t) = \xi(X_T) - \int_t^T f(s, X_s)\,\mathrm{d}s,$$

where X, Y are *deterministic* processes. In other words, $\frac{\mathrm{d}}{\mathrm{d}t} X_t = \mu(t, X_t)$, $\frac{\mathrm{d}}{\mathrm{d}t}\left[u(t, X_t)\right] = -f(t, X_t)$. If we look at X as a process which starts at X_T at time T and moves backwards in time much like Y, we see that u can be calculated along X starting at the value $\xi(X_T)$ and following the ODE $\frac{\mathrm{d}}{\mathrm{d}t}\left[u(t, X_t)\right] = -f(t, X_t)$. But this is equivalent to the so called method of characteristics for the first order linear PDE

$$u_t(t,x) + \sum_{i=1}^n \mu_i(t,x) u_{x_i}(t,x) = f(t,x), \qquad u(T, \cdot) = \tilde{\xi},$$

with X being the *characteristic curve* starting at x along which u is reduced to another ODE.
In that light, decoupling fields together with the underlying process structure X, Y, Z can be interpreted as a generalization of the method of characteristics for the aforementioned second order problems, i.e. for problems where σ does not vanish and second order partial derivatives appear in the PDE. The difference to the first order case is that the processes X, Y, Z can have non-deterministic random trajectories.
This perspective has important implications:

- While theoretical results for infinite-dimensional spaces like the Nash-Moser theorem or the theorem of Leray-Schauder are often used to obtain solutions in PDE theory, whose existence is provided implicitly, a more *explicit* construction using Picard iterations can be executed if the system is interpreted probabilistically as above: This is because, from a theoretical point of view, stochastic differential equations behave similarly to ordinary differential equations. So, roughly speaking, using the underlying probabilistic structure given by X, Y, Z in order to explain the dynamics of u reduces the "complexity" of the associated PDE to that of an ODE.

- The method of characteristics is particularly important from the point of view of numerics. This is due to the fact that numerical schemes for ODEs are often much faster than those for PDEs. So, reducing the problem to an ODE greatly simplifies its numerical treatment. In that light, the aforementioned probabilistic interpretation of the problem via decoupling fields is a promising starting point for numerical schemes.

Now, let us assume that σ does not vanish. For instance, assume that μ and σ are functions of s, X_s, while f is zero. Then the forward equation satisfied by X becomes a standard forward diffusion

$$X_t = x + \int_0^t \mu(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}W_s,$$

while $Y_\cdot = u(\cdot, X_\cdot)$ is a local martingale ending in $\xi(X_T)$. We have, thereby, recovered the notion of a *stochastic solution* to the associated parabolic PDE, which was proposed in [SV72] and further investigated in several recent works, for instance in [BS12] in connection with viscosity solutions. That concept is particularly useful when dealing with degenerate problems. In a way it allows to formulate dynamics of u without the use of partial derivatives. Instead, they are encoded through the stochastic process X.

Note, however, that this connection between decoupling fields and PDEs only makes sense in the Markovian case.

Let us finally remark that these considerations are closely tied to the famous Feynman-Kac formula, which under certain conditions allows to express u using conditional expectations of expressions where randomness is generated by Brownian noise.

1.5 Outlook

Several interesting questions arise in the context of Chapter 2 and its applications, which motivate additional research:

- In [HHI$^+$14] the problem of utility maximization in incomplete markets has been considered not just for the case of a utility function U defined on $\mathbb{R}$, but also for the case of U defined on the domain $(0, \infty)$. This second case leads to a different FBSDE, the solvability of which has not been investigated in this work. Instead, we have concentrated on the first case in Chapter 5 to demonstrate the technique. Yet we have no reason to believe that a similar analysis cannot be performed for the second case.
  Let us remark that for the case of power utility, which is defined on $(0, \infty)$ and belongs to the second category, the corresponding FBSDE has already been treated successfully in [Zha13] using convex duality in particular.

- While many natural phenomena can be described using PDEs of parabolic type and, thereby, also by FBSDEs with decoupling fields, some models rely on so called partial differential integral equations (PDIE), which have a more general structure due to the fact that integrals w.r.t. space appear together with partial derivatives in the equation. It is known that these problems are connected to BSDEs driven by general Lévy noise rather than more special Brownian noise (e.g. [NS01]). It would be, therefore, interesting and natural to investigate to what extent the method of decoupling fields can be generalized to coupled systems where Lévy noise is used.
  One can raise a similar question with regard to certain stochastic partial differential equations (SPDE) which are known to be connected to so called backward doubly stochastic differential equations (BDSDE), another generalization of (F)BSDE. Consult [PP94] for this link.

- In Chapter 2 we have implemented a rather explicit construction of the decoupling field u for a relatively general class of problems. We rely on constructing X, Y, Z locally using a Picard iteration together with Banach's fixed point theorem and "glueing" together decoupling fields defined on small intervals. Note that many existing numerical schemes use decoupling fields implicitly (without necessarily using this name), e.g. [BZ08], [DM08]. It is, therefore, natural to investigate the applicability of Chapter 2 to numerics of FBSDEs. Our analysis of higher order differentiability of u in Chapter 4 (for a special system) may find application in this context.

Also, our studies of uniform boundedness of Z in the MLLC case and boundedness of Y, Z in the case of SLLC are of importance in this context, since they allow truncation and, thereby, reduction to Lipschitzian problems.

Chapter 2

Existence, Uniqueness and Regularity of Decoupling Fields

A general FBSDE in a Brownian setting is a system of the form

$$X_t = x + \int_0^t \mu(s, X_s, Y_s, Z_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s, Y_s, Z_s)\,\mathrm{d}W_s,$$

$$Y_t = \xi(X_T) - \int_t^T f(s, X_s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s,$$

where X and Y can be $\mathbb{R}^n$ - resp. $\mathbb{R}^m$ - valued, such that the two equations represent systems of equations in general. The nature of the underlying problem is encoded in the parameter functions μ, σ, f which can be random, but are at least progressively measurable, and the terminal condition ξ, which can also depend on ω, but is required to be measurable w.r.t. $\mathcal{F}_T$, the information available at terminal time T.

As already mentioned the system is called decoupled if either μ, σ do not depend on Y, Z, or if ξ, f do not depend on X. In these two cases the problem can be treated by solving one of the equations first, and then simply plugging the solution processes obtained into the other equation, in order to solve the latter in the second step. In both steps solutions can be constructed through a Picard iteration applying Banach's fixed point theorem. Because of this, theory of decoupled problems is much more extensive than the theory of general strongly coupled FBSDE, in which still many questions remain open.

Strongly coupled problems can be ill-posed, i.e. solutions (X, Y, Z) might not exist. So it is natural to ask the question under which conditions the problem is actually well-posed, or what well-posedness precisely means. It also seems to make little sense to investigate ill-posed problems. However, our work actually contradicts this view in some sense. In this chapter we will drop the restriction of well-posedness and develop an existence, uniqueness and regularity theory for the general case. In other words we consider ill-posed problems no less interesting or worthy of a rigorous study than well-posed problems. We will merely require Lipschitz continuity of the parameters in the general non-Markovian case and a form of local Lipschitz continuity in the more special Markovian case. The terminal condition will always be Lipschitz continuous in the process variables. To accommodate the fact that we do not require well-posedness of the problem we introduce the so called *maximal interval*, which, roughly speaking, is the largest interval on which a given FBSDE system has reasonable solutions. The best case scenario is that the maximal interval coincides with $[0, T]$, such that the problem becomes well-posed. Based on our study of the form of the maximal interval and the behaviour of the decoupling field at the left boundary, in the ill-posed case, we will propose a general method to verify well-posedness via contradiction.

As already mentioned, we will work extensively with weak derivatives in this chapter. This might seem somewhat unusual since in the literature investigations of classical differentiability of X, Y, Z w.r.t the initial value are more common (e.g. [dR10]). However, for the general problems we consider this is problematic for the following reasons:

- We want to keep the basic theory as general as possible and seek to avoid assuming more than Lipschitz or local Lipschitz continuity in the parameters μ, σ, f, ξ.
- Every additional regularity assumed for ξ must also be shown for u locally in order to repeat the argument during concatenation. So, the more we assume the more we need to show putting the success of the whole construction at risk if the properties assumed are not inherited in the same form. At the same time assuming additional regularity for μ, σ, f without assuming the same for ξ makes little sense in light of the fact that we work with general possibly degenerate systems.

Moreover, we put special attention to various technical questions concerning decoupling fields, which are crucial not just as theoretical considerations but also for later applications. This includes:

- Analysis of measurability and continuity properties of the decoupling field u: e.g. progressive measurability, right-continuity or continuity (cf. Lemmas 2.1.3 and 2.5.15).
- Weak and strong regularity of u: e.g. expanding weak differentiability of u w.r.t. x into weak differentiability of X, Y, Z w.r.t. x (e.g. Corollary 2.5.4).
- Boundedness of Z in the general (strongly coupled and possibly degenerate) Markovian case (Corollary 2.5.24).
- Existence and uniqueness of X, Y, Z under local Lipschitz assumptions in the Markovian case.

This chapter is structured as follows. In Section 2.1 we will define the notion of a decoupling field and discuss some basic properties. Furthermore, we will summarize some basic results about weak derivatives. Although some of theses statements might seem to be straightforward, we have included their proofs in the Appendix, since we have not been able to find a proper source to cite.

In Section 2.2 we will prove a local existence and uniqueness result for decoupling fields (Theorem 2.2.1) for globally Lipschitz continuous coefficients. This result will serve as the basis for the theory developed thereafter. The proof is constructive and is based on a contractive Picard-Lindelöf iteration. In Section 2.3 we discuss two simple examples to motivate the hypotheses of Theorem 2.2.1. Section 2.4 deals with regularity properties of decoupling fields. As a by-product of the construction in Section 2.2 we will obtain variational differentiability of solutions. More precisely, we show that X, Y, Z depend in a weakly differentiable way on the initial condition $x \in \mathbb{R}^n$. In Section 2.5 we show global uniqueness and global regularity of decoupling fields, and study global existence by introducing the *maximal interval*. We will prove a necessary condition for the problem to be ill-posed, and propose a general method to verify well-posedness in those cases in which it is conjectured. Our approach is somewhat related to the study of the *characteristic BSDE* proposed in [MWZZ11].

Furthermore, we will discuss the Markovian case in more detail (Section 2.5.1). Here our theory can be extended to coefficients that are not globally Lipschitz continuous, a very useful feature for applications. The decoupling field also assumes nice properties in the Markovian case, such as being deterministic and continuous. We discuss the case in which the parameter functions are only locally Lipschitz in Y, Z. The case in which they are locally Lipschitz continuous in Z (and Lipschitz continuous in the remaining components) deserves separate consideration. This will be left for Chapter 4.

2.1 Preliminaries

2.1.1 Decoupling fields

We will consider families (μ, σ, f) of measurable functions, more precisely

$$\mu : [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \longrightarrow \mathbb{R}^n,$$
$$\sigma : [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \longrightarrow \mathbb{R}^{n\times d},$$
$$f : [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \longrightarrow \mathbb{R}^m,$$

where

- $n, m, d \in \mathbb{N}$ and $T > 0$,
- $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ is a complete filtered probability space,
- $\mathcal{F}_0 = \sigma(p) \vee \mathcal{N}$ for some $p : \Omega \to S$ and a polish space S, $\mathcal{N}$ are the null sets,
- $\mathcal{F}_t = \sigma(\mathcal{F}_0, (W_s)_{s\in[0,t]})$ holds, where $(W_t)_{t\in[0,T]}$ is a d-dimensional Brownian motion, independent of $\mathcal{F}_0$,
- $\mathcal{F} = \mathcal{F}_T$.

In particular, the filtration satisfies the usual conditions.
We want μ, σ and f to be progressively measurable w.r.t. $(\mathcal{F}_t)_{t\in[0,T]}$, i.e. $\mu\mathbf{1}_{[0,t]}, \sigma\mathbf{1}_{[0,t]}, f\mathbf{1}_{[0,t]}$ must be $\mathcal{B}([0,T]) \otimes \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^{m\times d})$ - measurable for all $t \in [0,T]$. As usual $\mathcal{B}([0,T])$, $\mathcal{B}(\mathbb{R}^n)$, etc. refers to the Borel σ - algebra on $[0,T]$, $\mathbb{R}^n$, etc. We will assume throughout the chapter that μ, σ and f have this property without mentioning it.

Definition 2.1.1. Let $\xi : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ be measurable and let $t \in [0,T]$.
We call a function $u : [t,T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ with $u(T, \omega, \cdot) = \xi(\omega, \cdot)$ for a.a. $\omega \in \Omega$ a *decoupling field* for $(\xi, (\mu, \sigma, f))$ on $[t,T]$ if for all $t_1, t_2 \in [t,T]$ with $t_1 \le t_2$ and any $\mathcal{F}_{t_1}$ - measurable $X_{t_1} : \Omega \to \mathbb{R}^n$ there exist progressive processes X, Y, Z on $[t_1, t_2]$ such that

- $X_s = X_{t_1} + \int_{t_1}^s \mu(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_{t_1}^s \sigma(r, X_r, Y_r, Z_r)\,\mathrm{d}W_r$ a.s.,
- $Y_s = Y_{t_2} - \int_s^{t_2} f(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^{t_2} Z_r\,\mathrm{d}W_r$ a.s.,
- $Y_s = u(s, X_s)$ a.s.,

for all $s \in [t_1, t_2]$. In particular, we want all integrals to be well-defined and X, Y, Z to have values in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$ respectively.

Some remarks about this definition:

- By well-defined we mean $\int_{t_1}^{t_2} |\mu(r, X_r, Y_r, Z_r)|\,\mathrm{d}r < \infty$ a.s., $\int_{t_1}^{t_2} |\sigma(r, X_r, Y_r, Z_r)|^2\,\mathrm{d}r < \infty$ a.s., etc.
- In the above definition the first equation is called the *forward equation*, the second the *backward equation* and the third will be referred to as the *decoupling condition*.
- Note that the forward equation implies in particular that the process X must satisfy

$$X(t_1, \cdot) = X_{t_1} \quad \text{a.s.}$$

where X_{t_1} is the given $\mathcal{F}_{t_1}$ - measurable random variable. By a slight abuse of notation we do not clearly distinguish between X_{t_1} and the random variable $X_s := X(s, \cdot)$ for $s = t_1$, because

they are a.s. equal anyway.
This requirement that X should start at X_{t_1} is referred to as the *initial condition*. By a slight abuse of notation we will sometimes refer to X_{t_1} itself as the initial condition. X_{t_1} is also sometimes referred to as the *initial value*.

- At this point, we do not require the triple (X, Y, Z) to be unique for given t_1, t_2, X_{t_1} or even unique up to modification.

- Note that if $t_2 = T$, we get $Y_T = \xi(X_T)$ a.s. as a consequence of the decoupling condition together with $u(T, \omega, \cdot) = \xi(\omega, \cdot)$ for a.a. $\omega \in \Omega$.

- If $t_2 = T$, we can say that a triple (X, Y, Z) solves the FBSDE on $[t_1, T]$, meaning that it satisfies the forward and the backward equation, together with $Y_T = \xi(X_T)$:

$$X_s = X_{t_1} + \int_{t_1}^{s} \mu(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_{t_1}^{s} \sigma(r, X_r, Y_r, Z_r)\,\mathrm{d}W_r,$$
$$Y_s = \xi(X_T) - \int_{s}^{T} f(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_{s}^{T} Z_r\,\mathrm{d}W_r, \quad \text{a.s. for all } s \in [t_1, T]. \quad (2.1)$$

The relationship $Y_T = \xi(X_T)$ is referred to as the *terminal condition*.
By an abuse of notation the function ξ itself is also sometimes referred to as the terminal condition and sometimes we will describe the relationship $u(T, \omega, \cdot) = \xi(\omega, \cdot)$ for a.a. $\omega \in \Omega$ with this term.
We might also at times refer to u as a decoupling field to FBSDE (2.1).

- The backward equation can equivalently be written in the form

$$Y_s = Y_{t_1} + \int_{t_1}^{s} f(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_{t_1}^{s} Z_r\,\mathrm{d}W_r \quad \text{a.s.} \quad \text{for all } s \in [t_1, t_2].$$

- One can also assume without loss of generality that X and Y are continuous in time: This is a direct consequence of the form of the forward and the backward equation and the fact that if we replace X, Y, Z with progressively measurable and continuous $\tilde{X}, \tilde{Y}, \tilde{Z}$ such that for every fixed $s \in [t_1, t_2]$ the equations $\tilde{X}_s = X_s$, $\tilde{Y}_s = Y_s$, $\tilde{Z}_s = Z_s$ hold almost surely, then the forward equation, the backward equation and the decoupling condition via u are still satisfied by $\tilde{X}, \tilde{Y}, \tilde{Z}$.

- We claim that $u(t_1, \cdot, x)$ is $\mathcal{F}_{t_1}$ - measurable for any $t_1 \in [t, T]$ and deterministic initial values $x \in \mathbb{R}^n$: This is a simple consequence of $u(t_1, \cdot, x) = Y_{t_1}$ a.s. and the adaptedness of Y.

- Under the assumption of continuity of X, Y it is obviously sufficient for the forward and the backward equations to be satisfied for all $s \in D$ for a countable dense subset D of $[t_1, t_2]$ as long as $t_1 \in D$.

- Remember that u is a function of (s, ω, x). The first component s is often referred to as the time and the last component $x \in \mathbb{R}^n$ as the space.

- If u is a decoupling field on an interval $[t, T]$, its restriction $u|_{[s,T]}$ to a subinterval $[s, T] \subseteq [t, T]$ is a decoupling field on $[s, T]$ for every such subinterval. By a slight abuse of notation we will simply say that u itself is a decoupling field on $[s, T]$.

Decoupling fields have the following very important property, which distinguishes them from classical solutions to FBSDEs. Its proof explains in particular why we work with general $\mathcal{F}_{t_1}$ - measurable random variables X_{t_1} in the definition of a decoupling field.

Lemma 2.1.2. *If u is a decoupling field for $(\xi,(\mu,\sigma,f))$ on $[t,T]$ and a map $\tilde{u}$ is a decoupling field for $(u(t,\cdot),(\mu,\sigma,f))$ on $[s,t]$, where $0\le s<t<T$, then the map*

$$\hat{u}:=\tilde{u}\mathbf{1}_{[s,t]}+u\mathbf{1}_{(t,T]}$$

is a decoupling field for $(\xi,(\mu,\sigma,f))$ on $[s,T]$.

Proof. Assume we have a $t_1\in[s,t)$ and a $t_2\in(t,T]$. Otherwise just apply the definition of a decoupling field for either $\tilde{u}$ or u.

Now, for any $\mathcal{F}_{t_1}$ - measurable $\hat{X}_{t_1}:\Omega\to\mathbb{R}^n$ we need to show existence of processes $\hat{X},\hat{Y},\hat{Z}$ solving our FBSDE on $[t_1,t_2]$ s.t. $\hat{Y}_r=\hat{u}(r,\hat{X}_r)$ a.s. for $r\in[t_1,t_2]$.

We construct theses processes in two steps: Firstly, we choose progressive processes $\tilde{X},\tilde{Y},\tilde{Z}$ on $[t_1,t]$ solving the FBSDE on $[t_1,t]$ with initial value $\hat{X}_{t_1}$ and terminal condition $\tilde{Y}_t=u(t,\tilde{X}_t)$ and fulfilling the decoupling condition $\tilde{Y}_r=\tilde{u}(r,\tilde{X}_r)=\hat{u}(r,\tilde{X}_r)$, $r\in[t_1,t]$, according to the definition of a decoupling field.
This gives us $\tilde{X}_t$. Now, using this random variable as a new initial condition we get in the second step progressive processes X,Y,Z on $[t,t_2]$ satisfying

- $X_r=\tilde{X}_t+\int_t^r\mu(v,X_v,Y_v,Z_v)\,\mathrm{d}v+\int_t^r\sigma(v,X_v,Y_v,Z_v)\,\mathrm{d}W_v$,
- $Y_r=Y_{t_2}-\int_r^{t_2}f(v,X_v,Y_v,Z_v)\,\mathrm{d}v-\int_r^{t_2}Z_v\,\mathrm{d}W_v$,
- $Y_r=u(r,X_r)=\hat{u}(r,X_r)$,

a.s. for all $r\in[t,t_2]$. Now, define $\hat{X}$ on $[t_1,t_2]$ via

$$\hat{X}:=\tilde{X}\mathbf{1}_{[t_1,t]}+X\mathbf{1}_{(t,t_2]}$$

and similarly define $\hat{Y}$ and $\hat{Z}$.
Note $\tilde{X}_t=X_t$ and also $\tilde{Y}_t=\tilde{u}(t,\tilde{X}_t)=u(t,X_t)=Y_t$ a.s. It is now easy to check that $\hat{X},\hat{Y},\hat{Z}$ satisfy the FBSDE on $[t_1,t_2]$ and the decoupling condition. We only check the forward equation:

- If $r\in[t_1,t]$, we have $\tilde{X}_r=\hat{X}_{t_1}+\int_{t_1}^r\mu(v,\tilde{X}_v,\tilde{Y}_v,\tilde{Z}_v)\,\mathrm{d}v+\int_{t_1}^r\sigma(v,\tilde{X}_v,\tilde{Y}_v,\tilde{Z}_v)\,\mathrm{d}W_v$, which due to the definition of $\hat{X}$, $\hat{Y}$, $\hat{Z}$ can be rewritten as

$$\hat{X}_r=\hat{X}_{t_1}+\int_{t_1}^r\mu(v,\hat{X}_v,\hat{Y}_v,\hat{Z}_v)\,\mathrm{d}v+\int_{t_1}^r\sigma(v,\hat{X}_v,\hat{Y}_v,\hat{Z}_v)\,\mathrm{d}W_v.$$

- If $r\in(t,t_2]$, we need to plug in $\tilde{X}_t=\hat{X}_{t_1}+\int_{t_1}^t\mu(v,\tilde{X}_v,\tilde{Y}_v,\tilde{Z}_v)\,\mathrm{d}v+\int_{t_1}^t\sigma(v,\tilde{X}_v,\tilde{Y}_v,\tilde{Z}_v)\,\mathrm{d}W_v$ into the forward equation for the process X in order to obtain

$$\begin{aligned}\hat{X}_r=X_r=\tilde{X}_t+\int_t^r\mu(v,X_v,Y_v,Z_v)\,\mathrm{d}v+\int_t^r\sigma(v,X_v,Y_v,Z_v)\,\mathrm{d}W_v=\\ =\hat{X}_{t_1}+\int_{t_1}^t\mu(v,\tilde{X}_v,\tilde{Y}_v,\tilde{Z}_v)\,\mathrm{d}v+\int_{t_1}^t\sigma(v,\tilde{X}_v,\tilde{Y}_v,\tilde{Z}_v)\,\mathrm{d}W_v+\\ +\int_t^r\mu(v,X_v,Y_v,Z_v)\,\mathrm{d}v+\int_t^r\sigma(v,X_v,Y_v,Z_v)\,\mathrm{d}W_v=\\ =\hat{X}_{t_1}+\int_{t_1}^r\mu(v,\hat{X}_v,\hat{Y}_v,\hat{Z}_v)\,\mathrm{d}v+\int_{t_1}^r\sigma(v,\hat{X}_v,\hat{Y}_v,\hat{Z}_v)\,\mathrm{d}W_v,\end{aligned}$$

  using the definition of $\hat{X}$, $\hat{Y}$, $\hat{Z}$.

□

Note that according to definition if u is a decoupling field and $\tilde{u}$ is a modification of u, i.e. for each $s \in [t,T]$ the functions $u(s,\omega,\cdot)$ and $\tilde{u}(s,\omega,\cdot)$ coincide for almost all $\omega \in \Omega$, then $\tilde{u}$ is also a decoupling field to the same problem. So, u could also be referred to as a class of modifications. Some of the representatives of the class might be progressively measurable, others not. We will see below that a progressively measurable representative does exist if the decoupling field is Lipschitz continuous in x:

Lemma 2.1.3. *Let $u : [t,T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ be a mapping such that*

- *u satisfies the properties of a decoupling field with the difference that only special initial conditions $X_{t_1} \in \mathcal{X}_{t_1}$ have to be considered, where $\mathcal{X}_{t_1}$ is some set of $\mathcal{F}_{t_1}$ - measurable $\mathbb{R}^n$ - valued random variables, which is defined for every $t_1 \in [t,T]$ and contains at least the constants $x \in \mathbb{R}^n$,*
- *$u(s,\cdot)$ is measurable for every $s \in [t,T]$,*
- *u is Lipschitz continuous in $x \in \mathbb{R}^n$ in the weak sense that there exists a constant $L > 0$ s.t. for every $s \in [t,T]$:*

$$|u(s,\omega,x) - u(s,\omega,x')| \leq L|x - x'| \qquad \forall x,x' \in \mathbb{R}^n, \quad \text{for a.a. } \omega \in \Omega.$$

Then u has a modification $\tilde{u}$ which is

- *progressively measurable,*
- *Lipschitz continuous in x in the strong sense*

$$|\tilde{u}(s,\omega,x) - \tilde{u}(s,\omega,x')| \leq L|x - x'| \qquad \forall s \in [t,T],\, \omega \in \Omega,\, x,x' \in \mathbb{R}^n$$

- *and "weakly right-continuous" in the sense that*

$$\lim_{k\to\infty} \tilde{u}(s_k,\cdot,X_k) = \tilde{u}(s',\cdot,X') \qquad a.s.,$$

for all $(s',X') \in [t,T] \times \mathcal{X}_{s'}$ and all sequences $(s_k) \subset [s',T]$, (X_k) where $X_k \in \mathcal{X}_{s_k}$, converging to s' and a.s. to X' respectively.

Proof. We can assume without loss of generality that u is truly Lipschitz continuous in x with Lipschitz constant L by modifying it for every fixed $s \in [t,T]$ such that $u(s,\omega,\cdot)$ is set to 0, if it is not Lipschitz continuous with constant L. This will have to be done for a set of ω, which has measure zero (for each fixed $s \in [t,T]$).

Choose any t_1 from the interval $[t,T]$ and some $X' \in \mathcal{X}_{t_1}$ as initial value. We have progressive processes X,Y,Z such that

- $X_s = X' + \int_{t_1}^s \mu(r,X_r,Y_r,Z_r)\,\mathrm{d}r + \int_{t_1}^s \sigma(r,X_r,Y_r,Z_r)\,\mathrm{d}W_r$,
- $Y_s = Y_T - \int_s^T f(r,X_r,Y_r,Z_r)\,\mathrm{d}r - \int_s^T Z_r\,\mathrm{d}W_r$,
- $Y_s = u(s,X_s)$, a.s. for every $s \in [t_1,T]$.

Choose any $t_2 \in [t_1,T]$ and $X'' \in \mathcal{X}_{t_2}$. We can assume that X and Y are continuous (we can choose such modifications). We use the triangle inequality together with the decoupling condition $Y_s = u(s,X_s)$:

$$|u(t_2,X'') - u(t_1,X')| \leq |u(t_2,X'') - u(t_2,X')| + |u(t_2,X') - u(t_2,X_{t_2})| + |u(t_2,X_{t_2}) - u(t_1,X')| \leq$$
$$= L|X'' - X'| + |u(t_2,X_{t_1}) - u(t_2,X_{t_2})| + |Y_{t_2} - Y_{t_1}| \leq L|X'' - X'| + L|X_{t_2} - X_{t_1}| + |Y_{t_2} - Y_{t_1}|.$$

Choosing sequences $t_2^{(k)} \downarrow t_1$ and $X^{(k)} \to X'$ a.s., where $X^{(k)} \in \mathcal{X}_{t_2^{(k)}}$ we obtain

$$\lim_{k\to\infty} u(t_2^{(k)}, \cdot, X^{(k)}) = u(t_1, \cdot, X') \qquad \text{a.s.} \tag{2.2}$$

from the continuity of the processes X and Y.

We claim that $u(t_1, \cdot, x')$ is $\mathcal{F}_{t_1}$ - measurable for any $t_1 \in [t, T]$ and deterministic $x' \in \mathbb{R}^n$: This is a simple consequence of $u(t_1, \cdot, x') = Y_{t_1}$ a.s. and the adaptedness of Y.

Now, define $\tilde{u}$ via

$$\tilde{u}(s, \omega, x) := \limsup_{k\to\infty} \sum_{l=1}^{k} u\left(t + l\frac{T-t}{k}, \omega, x\right) \mathbf{1}_{\left(t+(l-1)\frac{T-t}{k}, t+l\frac{T-t}{k}\right]}(s), \quad s \in [t, T], \omega \in \Omega, x \in \mathbb{R}^n$$

and observe:

- In the above sum for every (s, ω, x) there is at most one summand, which does not vanish.
- $\tilde{u}$ clearly inherits the (strong) Lipschitz continuity in x from u.
- For any $s \in [t, T]$ and any $\varepsilon > 0$ the function $\tilde{u}\mathbf{1}_{[t,s]}$ is $\mathcal{B}([0, T]) \otimes \mathcal{F}_{s+\varepsilon} \otimes \mathcal{B}(\mathbb{R}^n)$ - measurable, since
$$(s, \omega, x) \longmapsto u\left(t + l\frac{T-t}{k}, \omega, x\right) \mathbf{1}_{\left(t+(l-1)\frac{T-t}{k}, t+l\frac{T-t}{k}\right] \cap [t,s]}$$
is measurable w.r.t. this σ-algebra if k is large enough, such that $u\left(t + l\frac{T-t}{k}, \cdot, x\right)$ is measurable w.r.t. $\mathcal{F}_{t+l\frac{T-t}{k}} \subseteq \mathcal{F}_{s+\varepsilon}$ for $s \in \left(t + (l-1)\frac{T-t}{k}, t + l\frac{T-t}{k}\right]$.
Thus $\tilde{u}\mathbf{1}_{[t,s]}$ is $\mathcal{B}([0, T]) \otimes \mathcal{F}_{s+} \otimes \mathcal{B}(\mathbb{R}^n)$ - measurable for all s and so $\tilde{u}$ is progressively measurable due to $\mathcal{F}_{s+} = \mathcal{F}_s$.
- For all $s \in [t, T]$ and all $x \in \mathbb{R}^n$ the random variables $\tilde{u}(s, \cdot, x)$ and $u(s, \cdot, x)$ are a.s. equal, since $\lim_{k\to\infty} u\left(t + l(s, k)\frac{T-t}{k}, \omega, x\right) = u(s, \omega, x)$ for a.a. ω, where $l(s, k)$ is the unique element of $\{1, \ldots, k\}$ s.t. $\mathbf{1}_{\left(t+(l(s,k)-1)\frac{T-t}{k}, t+l(s,k)\frac{T-t}{k}\right]}(s) = 1$. Note here that $t + l(s, k)\frac{T-t}{k} \geq s$ converges to s for $k \to \infty$, so we can apply (2.2).
Due to Lipschitz continuity in x the maps $x \mapsto \tilde{u}(s, \omega, x)$ and $x \mapsto u(s, \omega, x)$ must also coincide for a.a. $\omega \in \Omega$:
They coincide for every fixed x up to a null set and so they a.s. coincide on some fixed countable dense subset of $\mathbb{R}^n$. Now, continuity implies that they a.s. coincide on the whole of $\mathbb{R}^n$.
- As a modification $\tilde{u}$ inherits the "weak right-continuity" of u, i.e. the property (2.2).

□

2.1.2 Weak derivatives

In this thesis we will work extensively with weak derivatives. This will allow us to show variational differentiability (i.e. w.r.t. the initial value $x \in \mathbb{R}^n$) of the processes X, Y, Z for Lipschitz continuous μ, σ, f, ξ.
We start by fixing notation and giving some definitions:

For the following $|\cdot|$ will denote the usual Euclidean norm in any finite dimensional Euclidean space.
We can interpret elements of $\mathbb{R}^{n\times d}$ and $\mathbb{R}^{m\times d}$ as matrices or as linear operators on $\mathbb{R}^d$ with values in $\mathbb{R}^n$ or $\mathbb{R}^m$. Similarly we interpret $\mathbb{R}^{m\times d\times n}$ as the space of linear mappings on $\mathbb{R}^n$ to $\mathbb{R}^{m\times d}$.

If $x \in \mathbb{R}^{m\times d}$ or $x \in \mathbb{R}^{n\times d}$, the expression $|x|$ denotes the Frobenius norm of the linear operator x, i.e. the square root of the sum of the squares of its matrix coefficients.

We denote by $S^{n-1} := \{x \in \mathbb{R}^n \,|\, |x| = 1\}$ the $(n-1)$ - dimensional sphere.

If $x \in \mathbb{R}^{n\times n}$ or $x \in \mathbb{R}^{m\times n}$ or $x \in \mathbb{R}^{m\times d\times n}$ or $x \in \mathbb{R}^{n\times d\times n}$, we define $|x|_v := |x \cdot v|$ for all $v \in S^{n-1}$, where $\cdot$ is the application of the linear operator x to the vector v such that $x \cdot v$ is in $\mathbb{R}^n$ or $\mathbb{R}^m$ or $\mathbb{R}^{m\times d}$ or $\mathbb{R}^{n\times d}$ respectively. We refer to $\sup_{v\in S^{n-1}} |x|_v$ as the operator norm of x.

For a measurable map $\xi : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ we define

$$L_{\xi,x} := \inf \left\{ L \geq 0 \,|\, \text{for a.a. } \omega \in \Omega : \ |\xi(\omega, x) - \xi(\omega, x')| \leq L|x - x'| \text{ for all } x, x' \in \mathbb{R}^n \right\},$$

where $\inf \emptyset := \infty$. We also set $L_{\xi,x} := \infty$ if ξ is not measurable. $L_{\xi,x} < \infty$ implies that ξ is Lipschitz continuous in x in some sense.

For a map $u : [t, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ we define

$$L_{u,x} := \sup_{s\in[t,T]} L_{u(s,\cdot),x}.$$

Now, consider a mapping $X : \mathcal{M} \times \Lambda \to \mathbb{R}$, where $(\mathcal{M}, \mathcal{A}, \rho)$ is some measure space with finite measure ρ and $\Lambda \subseteq \mathbb{R}^N$ is open, $N \in \mathbb{N}$. We say that X is *weakly differentiable* w.r.t. the parameter $\lambda \in \Lambda$, if for almost all $\omega \in \mathcal{M}$ the mapping $X(\omega, \cdot) : \Lambda \to \mathbb{R}$ is weakly differentiable. This means that there exists a mapping $\frac{\mathrm{d}}{\mathrm{d}\lambda} X : \mathcal{M} \times \Lambda \to \mathbb{R}^{1\times N}$ such that

$$\int_\Lambda \varphi(\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda} X(\omega, \lambda) \,\mathrm{d}\lambda = - \int_\Lambda X(\omega, \lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda} \varphi(\lambda) \,\mathrm{d}\lambda, \tag{2.3}$$

for any real valued test function $\varphi \in C_c^\infty(\Lambda)$, for almost all $\omega \in \mathcal{M}$. In particular, $X(\omega, \cdot)$ and the *weak derivative* $\frac{\mathrm{d}}{\mathrm{d}\lambda} X(\omega, \cdot)$ have to be locally integrable for a.a. ω. This of course includes measurability w.r.t. λ for almost every *fixed* ω.
Similarly we could also define higher order weak differentiability. Weak differentiability for vector valued mappings is defined component-wise.

We call two maps $Y, Z : \mathcal{M} \times \Lambda \to \mathbb{R}^{1\times N}$ versions of each other if $Y(\omega, \cdot)$ and $Z(\omega, \cdot)$ are a.e. equal for almost every fixed ω. Obviously a version of a weak derivative is again a weak derivative (of the same X).
If X is a *measurable* function of (ω, λ), its weak derivative $\frac{\mathrm{d}}{\mathrm{d}\lambda} X$ will have a measurable version: For all $v \in \mathbb{R}^N$ and all $h > 0$ we can write

$$\int_0^h \frac{\mathrm{d}}{\mathrm{d}\lambda} X(\omega, \lambda_0 + tv) v \,\mathrm{d}t = X(\omega, \lambda_0 + hv) - X(\omega, \lambda_0), \tag{2.4}$$

for a.a. $\lambda_0 \in \Lambda$, s.t. $\overline{B_{h|v|}(\lambda_0)} \subseteq \Lambda$, for almost every $\omega \in \mathcal{M}$ (Lemma A.2.1). For instance, choose $h = h_n = n^{-1}$, $n \in \mathbb{N}$. Clearly, $Y(\omega, \lambda_0) := \limsup_{n\to\infty} \frac{1}{h_n}(X(\omega, \lambda_0 + h_n v) - X(\omega, \lambda_0))$ is a measurable function of $(\omega, \lambda_0) \in \Omega \times \Lambda$. However, Y is a version of $\frac{\mathrm{d}}{\mathrm{d}\lambda} X v$ due to (2.4) and the fundamental Theorem of Lebesgue integral calculus. This allows us to construct a measurable version of $\frac{\mathrm{d}}{\mathrm{d}\lambda} X$ by taking canonical unit vectors for v.
The relationship $\frac{\mathrm{d}}{\mathrm{d}\lambda} X(\omega, \lambda_0) v = \limsup_{n\to\infty} \frac{1}{h_n}(X(\omega, \lambda_0 + h_n v) - X(\omega, \lambda_0))$, which holds for almost all λ_0, for almost all ω, also implies uniqueness of $\frac{\mathrm{d}}{\mathrm{d}\lambda} X$ up to different versions.

It is well-known that for $N = 1$ weakly differentiable functions $X(\omega, \cdot)$ have continuous versions: They are even absolutely continuous (after being amended on a null set, e.g. consult [Maz11], Section 1.1.3). However weak differentiability does not imply ω-wise continuity w.r.t. λ in general.

Also note that if X is weakly differentiable, any version of X will be weakly differentiable as well with the same weak derivative.

Note furthermore the difference between requiring the relationship (2.3) to hold for every φ for a.a. ω or to require it to hold for a.a. ω for every fixed $\varphi \in C_c^\infty(\Lambda)$. One can check that these definitions are equivalent by using only countably many $\varphi \in C_c^\infty(\Lambda)$, but choosing them in such a way that all other function in $C_c^\infty(\Lambda)$ (together with their derivatives) can be approximated arbitrarily well in the supremum norm by linear combinations of these functions (or of their derivatives respectively).

Remember the definition of $L_{\xi,x}$. We have the following characterization of this value:

Lemma 2.1.4. *A measurable map $\xi : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ satisfies $L_{\tilde{\xi},x} < \infty$ for some $\tilde{\xi} = \xi$ a.e. if and only if ξ is weakly differentiable w.r.t x such that $\frac{\mathrm{d}}{\mathrm{d}x}\xi : \Omega \times \mathbb{R}^n \to \mathbb{R}^{m\times n}$ is bounded up to a null set. Furthermore, in this case*

$$L_{\tilde{\xi},x} = \sup_{v\in S^{n-1}} \operatorname{ess\,sup}_{\omega\in\Omega,x\in\mathbb{R}^n} \left| \frac{\mathrm{d}}{\mathrm{d}x}\xi(\omega,x)\right|_v .$$

Proof. See Appendix, Section A.2. ☐

Note that we have the following "chain rule" for weak derivatives:

Lemma (Lemma A.3.2 in the Appendix)**.** *Let $g : \mathcal{M} \times \mathbb{R}^N \to \mathbb{R}^m$ be measurable and Lipschitz continuous in the second component, which is further divided via $\mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_k}$ into $k \in \mathbb{N}$ different components. Let L_{g,x_i} be the Lipschitz constant w.r.t. the i-th component for $i = 1, \dots, k$. Furthermore, let $X_i : \mathcal{M} \times \mathbb{R}^n \to \mathbb{R}^{N_i}$, $i = 1, \dots, k$ be measurable and weakly differentiable w.r.t. $\lambda \in \mathbb{R}^n$. So, $X := (X_1, \dots, X_k)^\top$ is $\mathbb{R}^N$-valued. Then the measurable mapping $g(X) : \mathcal{M} \times \mathbb{R}^n \to \mathbb{R}^m$ is also weakly differentiable w.r.t. $\lambda \in \mathbb{R}^n$ and, furthermore, there exist measurable mappings $\Delta^X_{x_i} g : \mathcal{M} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{m\times N_i}$ s.t.*

- $\sup_{w\in S^{N_i-1}} |\Delta^X_{x_i} g(\cdot,\cdot,\cdot)w| \le L_{g,x_i}$ *everywhere for every $i = 1, \dots, k$ and, moreover,*
- *for all $v \in \mathbb{R}^n$*

$$\left(\frac{\mathrm{d}}{\mathrm{d}\lambda} g(X)(\omega,\lambda)\right) v = \sum_{i=1}^k \left(\Delta^X_{x_i} g(\omega,\lambda,v)\right) \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} X_i(\omega,\lambda)\right) v$$

  holds for almost all $\lambda \in \mathbb{R}^n$, $\omega \in \mathcal{M}$.

Also, for later reference note Lemmas A.2.4 to A.2.8 in the appendix. They will be needed to justify interchanging differentiation (in the weak sense) with integration w.r.t. time, probability measure or Brownian motion.

Finally, we will need the following stability result in our construction. The statement should be familiar, yet we include the proof in the appendix for the sake of completeness and due to the randomized nature of the objects we work with.

Lemma 2.1.5. *Let $(\mathcal{M}, \mathcal{A}, \rho)$ be some measure space with finite measure ρ and let $\Lambda \subseteq \mathbb{R}^N$ be open. Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of measurable real valued maps on $\Lambda\times\mathcal{M}$ s.t. $X_i(\cdot,\omega)$ has all weak derivatives w.r.t. $\lambda \in \Lambda$ up to order $\delta \in \mathbb{N}$ and s.t. there exists a constant $C < \infty$ with*

$$\sum_{1\le|\alpha|\le\delta} \int_{\mathcal{M}} |D^\alpha_\lambda X_i(\lambda,\cdot)|^2 \,\mathrm{d}\rho \le C,$$

for almost all $\lambda \in \Lambda$ and all $i \in \mathbb{N}$, where $\alpha \in \mathbb{N}^N$ is a multi-index and $D^\alpha_\lambda X_i$ denotes the associated weak derivative w.r.t $\lambda \in \Lambda$.
Assume further that there exists a real valued map X on $\Lambda \times \mathcal{M}$ such that $\lim_{i\to\infty} X_i(\lambda,\cdot) = X(\lambda,\cdot)$ a.e. and in $\mathcal{L}^2(\mathcal{M})$ for all $\lambda \in \Lambda$.

Then there exists a measurable $\tilde{X}$ on $\Lambda \times \mathcal{M}$ with $\tilde{X}(\lambda, \cdot) = X(\lambda, \cdot)$ a.e. for all $\lambda \in \Lambda$ s.t. $\tilde{X}$ has all weak derivatives w.r.t. λ up to order δ and satisfies

$$\sum_{1 \leq |\alpha| \leq \delta} \int_{\mathcal{M}} |D^\alpha_\lambda \tilde{X}(\lambda, \cdot)|^2 \,\mathrm{d}\rho \leq C$$

for almost all $\lambda \in \Lambda$.
Furthermore, there exists a subsequence $(X_{i_k})_{k\in\mathbb{N}}$ of $(X_i)_{i\in\mathbb{N}}$ such that for every $\alpha \in \mathbb{N}^N$ with $1 \leq |\alpha| \leq \delta$ the sequence $(D^\alpha_\lambda X_{i_k})_{k\in\mathbb{N}}$ converges to $D^\alpha_\lambda \tilde{X}$ weakly in $\mathcal{L}^2(K \times \mathcal{M})$ for all compact $K \subset \Lambda$.

Proof. See Appendix, Section A.2. □

2.2 Local existence and uniqueness

We denote by $L_{\sigma,z}$ the Lipschitz constant of σ w.r.t. the dependence on the last component z (and w.r.t. the Frobenius norms on $\mathbb{R}^{m\times d}$ and $\mathbb{R}^{n\times d}$). We set $L_{\sigma,z} = \infty$ if σ is not Lipschitz continuous in z.
By $L^{-1}_{\sigma,z} = \frac{1}{L_{\sigma,z}}$ we mean $\frac{1}{L_{\sigma,z}}$ if $L_{\sigma,z} > 0$ and ∞ otherwise.

In the following we need further notation. For an integrable real valued random variable X the expression $\mathbb{E}_t[X]$ refers to $\mathbb{E}[X|\mathcal{F}_t]$, while $\mathbb{E}_{t,\infty}[X]$ refers to $\operatorname{ess\,sup} \mathbb{E}[X|\mathcal{F}_t]$, which might be ∞, but is always well-defined as the infimum of all constants $c \in [-\infty, \infty]$ such that $\mathbb{E}[X|\mathcal{F}_t] \leq c$ a.s..
As usual $\|X\|_\infty$ refers to the essential supremum of $|X|$, for an arbitrary measurable X.

Theorem 2.2.1. *Let*

- *μ, σ, f be Lipschitz continuous in (x, y, z) with Lipschitz constant L s.t.*
- *$\|(|\mu| + |f| + |\sigma|)(\cdot, \cdot, 0, 0, 0)\|_\infty < \infty$,*
- *$\xi : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ be measurable s.t. $\|\xi(\cdot, 0)\|_\infty < \infty$ and $L_{\xi,x} < L^{-1}_{\sigma,z}$.*

Then there exists a time $t \in [0, T)$ such that $(\xi, (\mu, \sigma, f))$ has a unique (up to modification) decoupling field u on $[t, T]$ with $L_{u,x} < L^{-1}_{\sigma,z}$ and $\sup_{s\in[t,T]} \|u(s, \cdot, 0)\|_\infty < \infty$.

Proof. Let for some $t \in [0, T)$, which will be specified later, $X_t : \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n$ be a $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{F}_t$ - measurable function s.t. $X_t(\cdot, \omega)$ is weakly differentiable for almost all $\omega \in \Omega$ and

$$\operatorname{ess\,sup}_{\lambda\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \mathbb{E}_{\hat{t},\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda} X_t(\lambda, \cdot)\right|_v^2\right] < \infty, \tag{2.5}$$

for some fixed $\hat{t} \in [0, t]$. Assume furthermore that $\mathbb{E}_{\hat{t},\infty}\left[|X_t(\lambda, \cdot)|^2\right] < \infty$ for all $\lambda \in \mathbb{R}^n$.

We want to solve the coupled FBSDE

- $X_s = X_t + \int_t^s \mu(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_t^s \sigma(r, X_r, Y_r, Z_r)\,\mathrm{d}W_r$,
- $Y_s = \xi(X_T) - \int_s^T f(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^T Z_r\,\mathrm{d}W_r$, a.s. for all $s \in [t, T]$ and all $\lambda \in \mathbb{R}^n$,

which means that X, Y, Z would be functions of λ, ω and s.
Let $\mathbb{G}_{\hat{t}}$ be the space of all progressive $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ - valued processes (X, Y, Z) on $[t, T] \times \Omega$ s.t.

$$\|(X, Y, Z)\|_w := \max\left(\sup_{s\in[t,T]} \sqrt{\mathbb{E}_{\hat{t},\infty}[|X_s|^2]},\ (1 + L_{\sigma,z}) \sup_{s\in[t,T]} \sqrt{\mathbb{E}_{\hat{t},\infty}[|Y_s|^2]},\right.$$
$$\left.(1 + L_{\sigma,z})\sqrt{\mathbb{E}_{\hat{t},\infty}\left[\int_t^T |Z_s|^2\,\mathrm{d}s\right]}\right) < \infty. \tag{2.6}$$

The choice of this particular norm $\|\cdot\|_w$ will become clear later.
Note that since (X,Y,Z) among others depends on the parameter λ, the norm $\|(X,Y,Z)\|_w$ is actually a real-valued function of $\lambda\in\mathbb{R}^n$.
Let $\mathbb{H}_{\hat{t}}$ be the space of all progressive mappings

$$(X,Y,Z):\mathbb{R}^n\times[t,T]\times\Omega\longrightarrow\mathbb{R}^{n\times n}\times\mathbb{R}^{m\times n}\times\mathbb{R}^{m\times d\times n}$$

such that

$$\|(X,Y,Z)\|_s:=\operatorname{ess\,sup}_{\lambda\in\mathbb{R}^n}\sup_{v\in S^{n-1}}\|(X(\lambda,\cdot)v,Y(\lambda,\cdot)v,Z(\lambda,\cdot)v)\|_w<\infty.$$

Until further notice let $\lambda\in\mathbb{R}^n$ be fixed but arbitrary:

For any $(X^0,Y^0,Z^0)\in\mathbb{G}_{\hat{t}}$ there are unique $(X^1,Y^1,Z^1)=F(X^0,Y^0,Z^0)\in\mathbb{G}_{\hat{t}}$ s.t.

$$X_s^1:=X_t+\int_t^s\mu(r,X_r^0,Y_r^0,Z_r^0)\,\mathrm{d}r+\int_t^s\sigma(r,X_r^0,Y_r^0,Z_r^0)\,\mathrm{d}W_r,$$

$$Y_s^1:=\xi(X_T^1)-\int_s^T f(r,X_r^1,Y_r^0,Z_r^0)\,\mathrm{d}r-\int_s^T(Z_r^1)\,\mathrm{d}W_r,$$

a.s. for all $s\in[t,T]$. This is a well-known consequence of the martingale representation theorem.
This relationship defines a mapping $F:\mathbb{G}_{\hat{t}}\to\mathbb{G}_{\hat{t}}$.
In the sequel we will check that F is a contraction w.r.t. $\|\cdot\|_w$ if t is close enough to T, depending on L, $L_{\sigma,z}$ and $L_{\xi,x}$:

Let $\tilde{X}_t$ have the same properties as X_t, in particular (2.5). Define

$$\|X_t-\tilde{X}_t\|_2:=\left(\mathbb{E}_{\hat{t},\infty}\left[\left|X_t(\lambda,\cdot)-\tilde{X}_t(\lambda,\cdot)\right|^2\right]\right)^{\frac{1}{2}}.$$

Let $\tilde{F}:\mathbb{G}_{\hat{t}}\to\mathbb{G}_{\hat{t}}$ be the mapping associated with $\tilde{X}_t$ as introduced above. Now, let

$$(X^0,Y^0,Z^0),(\tilde{X}^0,\tilde{Y}^0,\tilde{Z}^0)\in\mathbb{G}_{\hat{t}}$$

be arbitrary. Set $(X^1,Y^1,Z^1):=F(X^0,Y^0,Z^0),(\tilde{X}^1,\tilde{Y}^1,\tilde{Z}^1):=\tilde{F}(\tilde{X}^0,\tilde{Y}^0,\tilde{Z}^0)\in\mathbb{G}_{\hat{t}}$. We obviously have

$$X_s^1-\tilde{X}_s^1=X_t-\tilde{X}_t+\int_t^s\mu(r,X_r^0,Y_r^0,Z_r^0)-\mu(r,\tilde{X}_r^0,\tilde{Y}_r^0,\tilde{Z}_r^0)\,\mathrm{d}r+$$
$$+\int_t^s\sigma(r,X_r^0,Y_r^0,Z_r^0)-\sigma(r,\tilde{X}_r^0,\tilde{Y}_r^0,\tilde{Z}_r^0)\,\mathrm{d}W_r$$

and, therefore, using Minkowski inequality:

$$\left(\mathbb{E}_{\hat{t}}\left[\left|X_s^1-\tilde{X}_s^1\right|^2\right]\right)^{\frac{1}{2}}\leq\|X_t-\tilde{X}_t\|_2+\left(\mathbb{E}_{\hat{t}}\left[\left|\int_t^s\mu(r,X_r^0,Y_r^0,Z_r^0)-\mu(r,\tilde{X}_r^0,\tilde{Y}_r^0,\tilde{Z}_r^0)\,\mathrm{d}r\right|^2\right]\right)^{\frac{1}{2}}+$$
$$+\left(\mathbb{E}_{\hat{t}}\left[\left|\int_t^s\sigma(r,X_r^0,Y_r^0,Z_r^0)-\sigma(r,\tilde{X}_r^0,\tilde{Y}_r^0,\tilde{Z}_r^0)\,\mathrm{d}W_r\right|^2\right]\right)^{\frac{1}{2}},$$

where the last summand can be written as

$$\left(\mathbb{E}_{\hat{t}}\left[\int_t^s\left|\sigma(r,X_r^0,Y_r^0,Z_r^0)-\sigma(r,\tilde{X}_r^0,\tilde{Y}_r^0,\tilde{Z}_r^0)\right|^2\,\mathrm{d}r\right]\right)^{\frac{1}{2}}$$

using Itô isometry. Now, use

$$
\begin{aligned}
&\sigma(r, X_r^0, Y_r^0, Z_r^0) - \sigma(r, \tilde{X}_r^0, \tilde{Y}_r^0, \tilde{Z}_r^0) = \\
&\qquad = \Big(\sigma(r, X_r^0, Y_r^0, Z_r^0) - \sigma(r, \tilde{X}_r^0, Y_r^0, Z_r^0)\Big) + \Big(\sigma(r, \tilde{X}_r^0, Y_r^0, Z_r^0) - \sigma(r, \tilde{X}_r^0, \tilde{Y}_r^0, Z_r^0)\Big) + \\
&\qquad\qquad + \Big(\sigma(r, \tilde{X}_r^0, \tilde{Y}_r^0, Z_r^0) - \sigma(r, \tilde{X}_r^0, \tilde{Y}_r^0, \tilde{Z}_r^0)\Big)
\end{aligned}
$$

and a similar telescopic sum for μ together with the Lipschitz continuity of μ, σ to estimate the value $\left(\mathbb{E}_{\hat{t}}\left[\left|X_s^1 - \tilde{X}_s^1\right|^2\right]\right)^{\frac{1}{2}}$ from above by

$$
\begin{aligned}
&\|X_t - \tilde{X}_t\|_2 + L\left(\mathbb{E}_{\hat{t}}\left[\left(\int_t^s |X_r^0 - \tilde{X}_r^0| + |Y_r^0 - \tilde{Y}_r^0| + |Z_r^0 - \tilde{Z}_r^0| \,\mathrm{d}r\right)^2\right]\right)^{\frac{1}{2}} + \\
&\qquad + \left(\mathbb{E}_{\hat{t}}\left[\int_t^s \left(L|X_r^0 - \tilde{X}_r^0| + L|Y_r^0 - \tilde{Y}_r^0| + L_{\sigma,z}|Z_r^0 - \tilde{Z}_r^0|\right)^2 \,\mathrm{d}r\right]\right)^{\frac{1}{2}},
\end{aligned}
$$

which can by further estimated using Minkowski inequality by

$$
\begin{aligned}
&\|X_t - \tilde{X}_t\|_2 + L\left(\mathbb{E}_{\hat{t}}\left[\left(\int_t^s |X_r^0 - \tilde{X}_r^0| \,\mathrm{d}r\right)^2\right]\right)^{\frac{1}{2}} + L\left(\mathbb{E}_{\hat{t}}\left[\left(\int_t^s |Y_r^0 - \tilde{Y}_r^0| \,\mathrm{d}r\right)^2\right]\right)^{\frac{1}{2}} + \\
&\qquad + L\left(\mathbb{E}_{\hat{t}}\left[\left(\int_t^s |Z_r^0 - \tilde{Z}_r^0| \,\mathrm{d}r\right)^2\right]\right)^{\frac{1}{2}} + \\
&+ L\left(\mathbb{E}_{\hat{t}}\left[\int_t^s |X_r^0 - \tilde{X}_r^0|^2 \,\mathrm{d}r\right]\right)^{\frac{1}{2}} + L\left(\mathbb{E}_{\hat{t}}\left[\int_t^s |Y_r^0 - \tilde{Y}_r^0|^2 \,\mathrm{d}r\right]\right)^{\frac{1}{2}} + \\
&\qquad + L_{\sigma,z}\left(\mathbb{E}_{\hat{t}}\left[\int_t^s |Z_r^0 - \tilde{Z}_r^0|^2 \,\mathrm{d}r\right]\right)^{\frac{1}{2}}.
\end{aligned}
$$

Now, use Cauchy-Schwarz inequality and

$$
\mathbb{E}_{\hat{t}}\left[\int_t^s |X_r^0 - \tilde{X}_r^0|^2 \,\mathrm{d}r\right] = \int_t^s \mathbb{E}_{\hat{t}}\left[|X_r^0 - \tilde{X}_r^0|^2\right] \,\mathrm{d}r \le (s-t) \sup_{r \in [t,s]} \mathbb{E}_{\hat{t},\infty}\left[|X_r^0 - \tilde{X}_r^0|^2\right],
$$

etc. to estimate $\left(\mathbb{E}_{\hat{t}}\left[\left|X_s^1 - \tilde{X}_s^1\right|^2\right]\right)^{\frac{1}{2}}$ from above by

$$
\begin{aligned}
&\|X_t - \tilde{X}_t\|_2 + L\sqrt{s-t}\left(\mathbb{E}_{\hat{t}}\left[\int_t^s |X_r^0 - \tilde{X}_r^0|^2 \,\mathrm{d}r\right]\right)^{\frac{1}{2}} + L\sqrt{s-t}\left(\mathbb{E}_{\hat{t}}\left[\int_t^s |Y_r^0 - \tilde{Y}_r^0|^2 \,\mathrm{d}r\right]\right)^{\frac{1}{2}} + \\
&\qquad + L\sqrt{s-t}\left(\mathbb{E}_{\hat{t}}\left[\int_t^s |Z_r^0 - \tilde{Z}_r^0|^2 \,\mathrm{d}r\right]\right)^{\frac{1}{2}} + \\
&+ L\sqrt{s-t}\left(\sup_{r \in [t,T]} \mathbb{E}_{\hat{t},\infty}\left[|X_r^0 - \tilde{X}_r^0|^2\right]\right)^{\frac{1}{2}} + L\sqrt{s-t}\left(\sup_{r \in [t,T]} \mathbb{E}_{\hat{t},\infty}\left[|Y_r^0 - \tilde{Y}_r^0|^2\right]\right)^{\frac{1}{2}} + \\
&\qquad + L_{\sigma,z}\left(\mathbb{E}_{\hat{t},\infty}\left[\int_t^s |Z_r^0 - \tilde{Z}_r^0|^2 \,\mathrm{d}r\right]\right)^{\frac{1}{2}}.
\end{aligned}
$$

Using some additional simple estimates and recombining the terms afterwards, we can obtain an estimate which does not depend on ω and s in order to end up with:

$$\begin{aligned}\sup_{s\in[t,T]}\left(\mathbb{E}_{\hat{t},\infty}\left[\left|X_s^1-\tilde{X}_s^1\right|^2\right]\right)^{\frac{1}{2}} &\le \|X_t-\tilde{X}_t\|_2+\\ &+L\left(T-t+\sqrt{T-t}\right)\left(\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[|X_r^0-\tilde{X}_r^0|^2\right]}+\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[|Y_r^0-\tilde{Y}_r^0|^2\right]}\right)+\\ &+\left(L_{\sigma,z}+L\sqrt{T-t}\right)\left(\mathbb{E}_{\hat{t},\infty}\left[\int_t^T|Z_r^0-\tilde{Z}_r^0|^2\,\mathrm{d}r\right]\right)^{\frac{1}{2}}. \end{aligned}\tag{2.7}$$

Using $1\le 1+L_{\sigma,z}$ as well as $\frac{1}{1+L_{\sigma,z}}\le 1$ and the definition of $\|\cdot\|_w$:

$$\begin{aligned}\sup_{s\in[t,T]}\left(\mathbb{E}_{\hat{t},\infty}\left[\left|X_s^1-\tilde{X}_s^1\right|^2\right]\right)^{\frac{1}{2}} &\le \|X_t-\tilde{X}_t\|_2+L\left(T-t+\sqrt{T-t}\right)\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[|X_r^0-\tilde{X}_r^0|^2\right]}+\\ &+L\left(T-t+\sqrt{T-t}\right)(1+L_{\sigma,z})\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[|Y_r^0-\tilde{Y}_r^0|^2\right]}+\\ &+\frac{L_{\sigma,z}+L\sqrt{T-t}}{1+L_{\sigma,z}}(1+L_{\sigma,z})\left(\mathbb{E}_{\hat{t},\infty}\left[\int_t^T|Z_r^0-\tilde{Z}_r^0|^2\,\mathrm{d}r\right]\right)^{\frac{1}{2}}\le\\ &\le\|X_t-\tilde{X}_t\|_2+\left(2L\left(T-t+\sqrt{T-t}\right)+\frac{L_{\sigma,z}}{1+L_{\sigma,z}}+L\sqrt{T-t}\right)\left\|(X^0-\tilde{X}^0,Y^0-\tilde{Y}^0,Z^0-\tilde{Z}^0)\right\|_w. \end{aligned}\tag{2.8}$$

Note that the constant in front of $\left\|(X^0-\tilde{X}^0,Y^0-\tilde{Y}^0,Z^0-\tilde{Z}^0)\right\|_w$ converges to $\frac{L_{\sigma,z}}{1+L_{\sigma,z}}<1$ for $t\to T$.

Now, let us deduce a similar estimate for $|Y^1-\tilde{Y}^1|$ using the backward equation for $s\in[t,T]$:

$$Y_s^1-\tilde{Y}_s^1+\int_s^T(Z_r^1-\tilde{Z}_r^1)\,\mathrm{d}W_r=\xi(X_T^1)-\xi(\tilde{X}_T^1)-\int_s^T\left(f(r,X_r^1,Y_r^0,Z_r^0)-f(r,\tilde{X}_r^1,\tilde{Y}_r^0,\tilde{Z}_r^0)\right)\mathrm{d}r$$

and, therefore, the value

$$\begin{aligned}&\left(\mathbb{E}_{\hat{t}}\left[|Y_s^1-\tilde{Y}_s^1|^2\right]+\mathbb{E}_{\hat{t}}\left[\int_s^T|Z_r^1-\tilde{Z}_r^1|^2\,\mathrm{d}r\right]\right)^{\frac{1}{2}}=\\ &=\left(\mathbb{E}_{\hat{t}}\left[|Y_s^1-\tilde{Y}_s^1|^2\right]+\mathbb{E}_{\hat{t}}\left[\left|\int_s^T(Z_r^1-\tilde{Z}_r^1)\,\mathrm{d}W_r\right|^2\right]\right)^{\frac{1}{2}}=\\ &=\left(\mathbb{E}_{\hat{t}}\left[\left|Y_s^1-\tilde{Y}_s^1+\int_s^T(Z_r^1-\tilde{Z}_r^1)\,\mathrm{d}W_r\right|^2\right]\right)^{\frac{1}{2}}\end{aligned}$$

can be estimated by

$$\left(\mathbb{E}_{\hat{t}}\left[|\xi(X_T^1)-\xi(\tilde{X}_T^1)|^2\right]\right)^{\frac{1}{2}}+\left(\mathbb{E}_{\hat{t}}\left[\left|\int_s^T f(r,X_r^1,Y_r^0,Z_r^0)-f(r,\tilde{X}_r^1,\tilde{Y}_r^0,\tilde{Z}_r^0)\,\mathrm{d}r\right|^2\right]\right)^{\frac{1}{2}},$$

where we used Minkowski inequality. Using $L_{\xi,x}<\infty$ and the Lipschitz continuity of f we can similarly to previous estimates control the above value by

$$L_{\xi,x}\left(\mathbb{E}_{\hat{t}}\left[|X_T^1-\tilde{X}_T^1|^2\right]\right)^{\frac{1}{2}}+L\,(T-t)\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[|X_r^1-\tilde{X}_r^1|^2\right]}+$$

$$+L\,(T-t)\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[|Y_r^0-\tilde{Y}_r^0|^2\right]}+L\sqrt{T-t}\left(\mathbb{E}_{\hat{t},\infty}\left[\int_t^T|Z_r^0-\tilde{Z}_r^0|^2\,\mathrm{d}r\right]\right)^{\frac{1}{2}}.$$

Now, use $\mathbb{E}_{\hat{t}}\left[|X_T^1-\tilde{X}_T^1|^2\right]\leq\sup_{r\in[t,T]}\mathbb{E}_{\hat{t},\infty}\left[|X_r^1-\tilde{X}_r^1|^2\right]$ to estimate this further from above by the value

$$(L_{\xi,x}+L(T-t))\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[|X_r^1-\tilde{X}_r^1|^2\right]}+L\,(T-t)\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[|Y_r^0-\tilde{Y}_r^0|^2\right]}+$$

$$+L\sqrt{T-t}\left(\mathbb{E}_{\hat{t},\infty}\left[\int_t^T|Z_r^0-\tilde{Z}_r^0|^2\,\mathrm{d}r\right]\right)^{\frac{1}{2}}. \quad (2.9)$$

Now, plug in the inequality of (2.7) into the above expression and regroup the terms to obtain

$$\left(\mathbb{E}_{\hat{t}}\left[|Y_s^1-\tilde{Y}_s^1|^2\right]\right)^{\frac{1}{2}}\vee\left(\mathbb{E}_{\hat{t}}\left[\int_s^T|Z_r^1-\tilde{Z}_r^1|^2\,\mathrm{d}r\right]\right)^{\frac{1}{2}}\leq$$

$$\leq\left(\mathbb{E}_{\hat{t}}\left[|Y_s^1-\tilde{Y}_s^1|^2\right]+\mathbb{E}_{\hat{t}}\left[\int_s^T|Z_r^1-\tilde{Z}_r^1|^2\,\mathrm{d}r\right]\right)^{\frac{1}{2}}\leq(L_{\xi,x}+L(T-t))\|X_t-\tilde{X}_t\|_2+$$

$$+(L_{\xi,x}+L(T-t))L\left(T-t+\sqrt{T-t}\right)\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[|X_r^0-\tilde{X}_r^0|^2\right]}+$$

$$+\left((L_{\xi,x}+L(T-t))L\left(T-t+\sqrt{T-t}\right)+L\,(T-t)\right)\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[|Y_r^0-\tilde{Y}_r^0|^2\right]}+$$

$$+\left((L_{\xi,x}+L(T-t))\left(L_{\sigma,z}+L\sqrt{T-t}\right)+L\sqrt{T-t}\right)\left(\mathbb{E}_{\hat{t},\infty}\left[\int_t^T|Z_r^0-\tilde{Z}_r^0|^2\,\mathrm{d}r\right]\right)^{\frac{1}{2}}.$$

Multiplying both sides with $1+L_{\sigma,z}$ and taking the essential supremum on the left side of "$\leq$" we finally obtain

$$\left\{(1+L_{\sigma,z})\left(\sup_{s\in[t,T]}\mathbb{E}_{\hat{t},\infty}\left[|Y_s^1-\tilde{Y}_s^1|^2\right]\right)^{\frac{1}{2}}\right\}\vee\left\{(1+L_{\sigma,z})\left(\mathbb{E}_{\hat{t},\infty}\left[\int_t^T|Z_r^1-\tilde{Z}_r^1|^2\,\mathrm{d}r\right]\right)^{\frac{1}{2}}\right\}\leq$$

$$\leq(1+L_{\sigma,z})(L_{\xi,x}+L(T-t))\|X_t-\tilde{X}_t\|_2+$$

$$+(1+L_{\sigma,z})(L_{\xi,x}+L(T-t))L\left(T-t+\sqrt{T-t}\right)\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[|X_r^0-\tilde{X}_r^0|^2\right]}+$$

$$+\left((L_{\xi,x}+L(T-t))L\left(T-t+\sqrt{T-t}\right)+L\,(T-t)\right)(1+L_{\sigma,z})\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[|Y_r^0-\tilde{Y}_r^0|^2\right]}+$$

$$+\left((L_{\xi,x}+L(T-t))\left(L_{\sigma,z}+L\sqrt{T-t}\right)+L\sqrt{T-t}\right)(1+L_{\sigma,z})\left(\mathbb{E}_{\hat{t},\infty}\left[\int_t^T|Z_r^0-\tilde{Z}_r^0|^2\,\mathrm{d}r\right]\right)^{\frac{1}{2}},$$

which using the definition of $\|\cdot\|_w$ can be controlled by

$$(1+L_{\sigma,z})(L_{\xi,x}+L(T-t))\|X_t-\tilde{X}_t\|_2+\Bigg((1+L_{\sigma,z})(L_{\xi,x}+L(T-t))L\left(T-t+\sqrt{T-t}\right)+$$
$$+\left((L_{\xi,x}+L(T-t))L\left(T-t+\sqrt{T-t}\right)+L\left(T-t\right)\right)+$$
$$+\left((L_{\xi,x}+L(T-t))\left(L_{\sigma,z}+L\sqrt{T-t}\right)+L\sqrt{T-t}\right)\Bigg)\left\|(X^0-\tilde{X}^0,Y^0-\tilde{Y}^0,Z^0-\tilde{Z}^0)\right\|_w. \quad (2.10)$$

Note that the constant in front of $\left\|(X^0-\tilde{X}^0,Y^0-\tilde{Y}^0,Z^0-\tilde{Z}^0)\right\|_w$ converges from above to the value $L_{\sigma,z}\cdot L_{\xi,x}<1$ for $t\to T$.

Combining (2.8) and (2.10) we have finally shown

$$\left\|(X^1-\tilde{X}^1,Y^1-\tilde{Y}^1,Z^1-\tilde{Z}^1)\right\|_w\le\beta_t\|X_t-\tilde{X}_t\|_2+\gamma_t\left\|(X^0-\tilde{X}^0,Y^0-\tilde{Y}^0,Z^0-\tilde{Z}^0)\right\|_w, \quad (2.11)$$

where $\beta_t:=\max\{1,(1+L_{\sigma,z})(L_{\xi,x}+L(T-t))\}$ and

$$\gamma_t:=\left(2\cdot L\left(T-t+\sqrt{T-t}\right)+\frac{L_{\sigma,z}}{1+L_{\sigma,z}}+L\sqrt{T-t}\right)\vee$$
$$\vee\Bigg\{(1+L_{\sigma,z})(L_{\xi,x}+L(T-t))L\left(T-t+\sqrt{T-t}\right)+$$
$$+\left((L_{\xi,x}+L(T-t))L\left(T-t+\sqrt{T-t}\right)+L\left(T-t\right)\right)+$$
$$+\left((L_{\xi,x}+L(T-t))\left(L_{\sigma,z}+L\sqrt{T-t}\right)+L\sqrt{T-t}\right)\Bigg\} \quad (2.12)$$

Note that $\gamma_t<1$ for $t<T$ large enough. More precisely, $\lim_{t\uparrow T}\gamma_t=\frac{L_{\sigma,z}}{1+L_{\sigma,z}}\vee(L_{\sigma,z}\cdot L_{\xi,x})$. Also, note that $\gamma:[0,T]\to[0,\infty)$ is decreasing and continuous.

If $\gamma_t<1$ holds, we can perform an iteration by setting $(X^0,Y^0,Z^0):=(0,0,0)$ and defining recursively

$$(X^k,Y^k,Z^k):=F(X^{k-1},Y^{k-1},Z^{k-1}),$$

for $k\in\mathbb{N}$. In particular

$$X_s^1:=X_t+\int_t^s\mu(r,0,0,0)\,\mathrm{d}r+\int_t^s\sigma(r,0,0,0)\,\mathrm{d}W_r,$$

$$Y_s^1:=\xi(X_T^1)-\int_s^T f(r,X_r^1,0,0)\,\mathrm{d}r-\int_s^T(Z_r^1)\,\mathrm{d}W_r.$$

According to Banach's fixed point theorem this sequence converges in $\mathbb{G}_{\hat{t}}$ to a fixed point of F, which is unique. This already shows existence and uniqueness of a $\mathbb{G}_{\hat{t}}$ - solution (X,Y,Z) of the considered coupled FBSDE for a sufficiently small interval $[t,T]$. This is because a fixed point of F is a $\mathbb{G}_{\hat{t}}$ - solution to the FBSDE and vice versa.
(X,Y,Z) is unique in the sense that a different solution $(X',Y',Z')\in\mathbb{G}_{\hat{t}}$ would satisfy $X_s=X_s'$, $Y_s=Y_s'$ a.s. for all $s\in[t,T]$ and $Z=Z'$ a.e.

Additionally, due to a priori estimates of the Banach fixed point theorem we have

$$\|(X,Y,Z)\|_w \leq \frac{1}{1-\gamma_t}\left\|(X^1-X^0,Y^1-Y^0,Z^1-Z^0)\right\|_w = \frac{1}{1-\gamma_t}\left\|(X^1,Y^1,Z^1)\right\|_w, \tag{2.13}$$

which in turn can be controlled by a bound which depends on

- γ_t,
- $\|X_t(\lambda,\cdot)\|_2$, $\|(|\mu|+|f|+|\sigma|)(\cdot,\cdot,0,0,0)\|_\infty$, $\|\xi(\cdot,0)\|_\infty$, $L_{\xi,x}$, L, T

and is monotonically increasing in these values:
To see this first use the above "forward equation" which defines X^1 to estimate $\sup_{s\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[|X_s^1|^2\right]}$ through the values

$$\|X_t(\lambda,\cdot)\|_2, \quad \|\mu(\cdot,\cdot,0,0,0)\|_\infty, \quad T \quad \text{and} \quad \|\sigma(\cdot,\cdot,0,0,0)\|_\infty$$

and then use this estimate for the "backward equation" which defines Y^1, Z^1 to obtain a control for $\sup_{s\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[|Y_s^1|^2\right]}$ and $\left(\mathbb{E}_{\hat{t},\infty}\left[\int_t^T |Z_r^1|^2\,\mathrm{d}r\right]\right)^{\frac{1}{2}}$ via

$$|\xi(X_T^1)| \leq |\xi(0)| + L_{\xi,x}|X_T^1| \text{ a.s. and } |f(r,X_r^1,0,0)| \leq |f(r,0,0,0)| + L|X_r^1|.$$

Furthermore, according to (2.11) we have

$$\left\|(X^k-X,Y^k-Y,Z^k-Z)\right\|_w \leq \gamma_t^k\left\|(X^0-X,Y^0-Y,Z^0-Z)\right\|_w$$

which due to $\gamma_t<1$ and a standard application of Borel-Cantelli-Lemma implies that for $k\to\infty$ the processes X^k,Y^k,Z^k converge not just in L^2 but also almost everywhere to X,Y,Z respectively and also X_s^k,Y_s^k converge a.s. to X_s,Y_s for every fixed $s\in[t,T]$. ✓

Let us now view $\lambda\in\mathbb{R}^n$ as a running variable again:

We claim that

- (X,Y,Z) is a progressively measurable function of (λ,s,ω) and even weakly differentiable w.r.t. λ such that
- X_s,Y_s are measurable functions of (λ,ω) for every fixed $s\in[t,T]$ and weakly differentiable w.r.t. λ such that
- $\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X, \frac{\mathrm{d}}{\mathrm{d}\lambda}Y, \frac{\mathrm{d}}{\mathrm{d}\lambda}Z\right)$ is in $\mathbb{H}_{\hat{t}}$.

To demonstrate this we again define $(X^0,Y^0,Z^0):=(0,0,0)$ and recursively

$$(X^k,Y^k,Z^k) := F(X^{k-1},Y^{k-1},Z^{k-1}),$$

$k\in\mathbb{N}$. We claim that for all $k\in\mathbb{N}_0$

- X^k,Y^k,Z^k are progressively measurable and weakly differentiable w.r.t. λ such that
- X_s^k,Y_s^k are measurable functions of (λ,ω) for every fixed $s\in[t,T]$ and weakly differentiable w.r.t. λ such that
- $\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X^k, \frac{\mathrm{d}}{\mathrm{d}\lambda}Y^k, \frac{\mathrm{d}}{\mathrm{d}\lambda}Z^k\right)\in\mathbb{H}_{\hat{t}}$:

Clearly, this holds for the index $k=0$. In order to implement an inductive argument, assume that it holds up to an index $k-1$. We need to show that it also holds for k. In order to do this we consider $(X^k,Y^k,Z^k)=F(X^{k-1},Y^{k-1},Z^{k-1})$, which is actually a system of two equations according to the definition of F. Now, differentiate it w.r.t. the parameter $\lambda\in\mathbb{R}^n$:
Using Lemmas A.2.5, A.2.6 and A.3.2 we obtain from the "forward equation" for all $v\in S^{n-1}$:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}X_s^k v=\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t v+\int_t^s \Delta_x^{(\dots)}\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^{k-1}v+\Delta_y^{(\dots)}\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_r^{k-1}v+\Delta_z^{(\dots)}\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v\,\mathrm{d}r+$$
$$+\int_t^s \Delta_x^{(\dots)}\sigma\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^{k-1}v+\Delta_y^{(\dots)}\sigma\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_r^{k-1}v+\Delta_z^{(\dots)}\sigma\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v\,\mathrm{d}W_r, \quad (2.14)$$

where "$(\dots)$" stands for $(r,X_r^{k-1},Y_r^{k-1},Z_r^{k-1})$.
The integrability conditions in the Lemmas used are satisfied due to $\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X^{k-1},\frac{\mathrm{d}}{\mathrm{d}\lambda}Y^{k-1},\frac{\mathrm{d}}{\mathrm{d}\lambda}Z^{k-1}\right)$ belonging to $\mathbb{H}_{\hat{t}}$.
The objects $\Delta_x^{(\dots)}\mu$, $\Delta_y^{(\dots)}\mu$, etc. are supplied by Lemma A.3.2 and might depend on $v\in S^{n-1}$ itself, which will not be a problem however. These objects are progressively measurable and uniformly bounded.

From (2.14) we deduce using Cauchy-Schwarz' and Minkowski's inequalities, as well as Itô's isometry:

$$\left(\mathbb{E}_{\hat{t}}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_s^k\right|_v^2\right]\right)^{\frac{1}{2}}\le\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}+L\left(T-t+\sqrt{T-t}\right)\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^{k-1}\right|_v^2\right]}+$$
$$+L\left(T-t+\sqrt{T-t}\right)\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_r^{k-1}\right|_v^2\right]}+$$
$$+\left(L_{\sigma,z}+L\sqrt{T-t}\right)\left(\mathbb{E}_{\hat{t},\infty}\left[\int_t^T\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}\right|_v^2\mathrm{d}r\right]\right)^{\frac{1}{2}}, \quad (2.15)$$

where $\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}:=\operatorname{ess\,sup}_{\lambda\in\mathbb{R}^n}\sup_{v\in S^{n-1}}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t(\lambda,\cdot)\right|_v^2\right]}$:
The arguments for this are analogous to the ones used to deduce (2.7). We also use the inequality $\sup_{w\in S^{m\times d-1}}|\Delta_z^{(\dots)}\sigma|_w\le L_{\sigma,z}$ provided by Lemma A.3.2 and similar estimates. Note that such a relationship implies

$$\left|\Delta_z^{(\dots)}\sigma\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v\right|=\left|\Delta_z^{(\dots)}\sigma\right|_{\frac{1}{\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v\right|}\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v}\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v\right|\le L_{\sigma,z}\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v\right|.$$

Now, similarly to (2.8) we can proceed to

$$\sup_{s\in[t,T]}\left(\mathbb{E}_{\hat{t},\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_s^k\right|_v^2\right]\right)^{\frac{1}{2}}\le\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}+L\left(T-t+\sqrt{T-t}\right)\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^{k-1}\right|_v^2\right]}+$$
$$+L\left(T-t+\sqrt{T-t}\right)(1+L_{\sigma,z})\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_r^{k-1}\right|_v^2\right]}+$$
$$+\frac{L_{\sigma,z}+L\sqrt{T-t}}{1+L_{\sigma,z}}(1+L_{\sigma,z})\left(\mathbb{E}_{\hat{t},\infty}\left[\int_t^T\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}\right|_v^2\mathrm{d}r\right]\right)^{\frac{1}{2}},$$

which in turn is estimated by the value

$$\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}+\left(2L\left(T-t+\sqrt{T-t}\right)+\frac{L_{\sigma,z}}{1+L_{\sigma,z}}+L\sqrt{T-t}\right)\left\|\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X^{k-1},\frac{\mathrm{d}}{\mathrm{d}\lambda}Y^{k-1},\frac{\mathrm{d}}{\mathrm{d}\lambda}Z^{k-1}\right)\right\|_s\leq$$

$$\leq\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}+\gamma_t\left\|\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X^{k-1},\frac{\mathrm{d}}{\mathrm{d}\lambda}Y^{k-1},\frac{\mathrm{d}}{\mathrm{d}\lambda}Z^{k-1}\right)\right\|_s. \tag{2.16}$$

Now, let us look at the "backward equation". Using Lemmas A.2.5, A.2.8 and A.3.2 we get for all $v\in S^{n-1}$:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s^k v+\int_s^T\frac{\mathrm{d}}{\mathrm{d}\lambda}(Z_r^k v)\,\mathrm{d}W_r=\Delta_x^{X_T^k}\xi\frac{\mathrm{d}}{\mathrm{d}\lambda}X_T^k v-$$

$$-\int_s^T\Delta_x^{(\dots)}f\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^k v+\Delta_y^{(\dots)}f\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_r^{k-1}v+\Delta_z^{(\dots)}f\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v\,\mathrm{d}r. \tag{2.17}$$

Here "$(\dots)$" stands for $(r,X_r^k,Y_r^{k-1},Z_r^{k-1})$. The local integrability conditions in the Lemmas used are satisfied due to $\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X^{k-1},\frac{\mathrm{d}}{\mathrm{d}\lambda}Y^{k-1},\frac{\mathrm{d}}{\mathrm{d}\lambda}Z^{k-1}\right)$ being in $\mathbb{H}_{\hat{t}}$ and $\operatorname{ess\,sup}_{\lambda\in\mathbb{R}^n}\sup_{s\in[t,T]}\mathbb{E}_{\hat{t},\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_s^k\right|_v^2\right]$ being finite as shown above.
Equation (2.17) implies

$$\left(\mathbb{E}_{\hat{t}}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s^k\right|_v^2\right]+\mathbb{E}_{\hat{t}}\left[\int_s^T\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^k\right|_v^2\mathrm{d}r\right]\right)^{\frac{1}{2}}\leq$$

$$\leq(L_{\xi,x}+L(T-t))\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^k\right|_v^2\right]}+L\,(T-t)\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_r^{k-1}\right|_v^2\right]}+$$

$$+L\sqrt{T-t}\left(\mathbb{E}_{\hat{t},\infty}\left[\int_t^T\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}\right|_v^2\mathrm{d}r\right]\right)^{\frac{1}{2}}.$$

The arguments used here are analogous to the ones we used at (2.9). We also use the inequality $\sup_{w\in S^{n-1}}\left|\Delta_x^{X_T^k}\xi\right|_w\leq L_{\xi,x}$ and similar estimates provided by Lemma A.3.2.
Now, using (2.15) and then recombining the terms leads to:

$$\left(\mathbb{E}_{\hat{t}}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s^k\right|_v^2\right]+\mathbb{E}_{\hat{t}}\left[\int_s^T\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^k\right|_v^2\mathrm{d}r\right]\right)^{\frac{1}{2}}\leq(L_{\xi,x}+L(T-t))\cdot\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}+$$

$$+(L_{\xi,x}+L(T-t))L\left(T-t+\sqrt{T-t}\right)\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^{k-1}\right|_v^2\right]}+$$

$$+\left((L_{\xi,x}+L(T-t))L\left(T-t+\sqrt{T-t}\right)+L\,(T-t)\right)\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t},\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_r^{k-1}\right|_v^2\right]}+$$

$$+\left((L_{\xi,x}+L(T-t))\left(L_{\sigma,z}+L\sqrt{T-t}\right)+L\sqrt{T-t}\right)\left(\mathbb{E}_{\hat{t},\infty}\left[\int_t^T\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}\right|_v^2\mathrm{d}r\right]\right)^{\frac{1}{2}}.$$

Using the definition of $\|\cdot\|_s$ we can further estimate this by the value

$$(L_{\xi,x}+L(T-t))\cdot\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}+\frac{1}{1+L_{\sigma,z}}\Bigg((1+L_{\sigma,z})(L_{\xi,x}+L(T-t))L\left(T-t+\sqrt{T-t}\right)+$$

$$+\left((L_{\xi,x}+L(T-t))L\left(T-t+\sqrt{T-t}\right)+L\left(T-t\right)\right)+$$

$$+\left((L_{\xi,x}+L(T-t))\left(L_{\sigma,z}+L\sqrt{T-t}\right)+L\sqrt{T-t}\right)\right)\left\|\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X^{k-1},\frac{\mathrm{d}}{\mathrm{d}\lambda}Y^{k-1},\frac{\mathrm{d}}{\mathrm{d}\lambda}Z^{k-1}\right)\right\|_s\leq$$

$$\leq(L_{\xi,x}+L(T-t))\cdot\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}+\frac{1}{1+L_{\sigma,z}}\cdot\gamma_t\left\|\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X^{k-1},\frac{\mathrm{d}}{\mathrm{d}\lambda}Y^{k-1},\frac{\mathrm{d}}{\mathrm{d}\lambda}Z^{k-1}\right)\right\|_s.$$

Multiplying both sides of the resulting estimate with $1+L_{\sigma,z}$ and combining it with the estimate at (2.16) finally yields

$$\left\|\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X^k,\frac{\mathrm{d}}{\mathrm{d}\lambda}Y^k,\frac{\mathrm{d}}{\mathrm{d}\lambda}Z^k\right)\right\|_s\leq$$

$$\leq\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}\vee\left((1+L_{\sigma,z})(L_{\xi,x}+L(T-t))\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}\right)+\gamma_t\left\|\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X^{k-1},\frac{\mathrm{d}}{\mathrm{d}\lambda}Y^{k-1},\frac{\mathrm{d}}{\mathrm{d}\lambda}Z^{k-1}\right)\right\|_s.$$

This completes the inductive argument. ✓

As mentioned $\gamma_t\leq\gamma_{t'}<1$ for all $t\in[t',T]$, if $t'<T$ is large enough. More precisely, $\lim_{t\uparrow T}\gamma_t=\frac{L_{\sigma,z}}{1+L_{\sigma,z}}\vee(L_{\sigma,z}\cdot L_{\xi,x})<1$. Choose $t'\in[0,T)$ as the smallest value in $[0,T]$ such that

$$1-\gamma_{t'}\geq\frac{1}{2}\left(1-\frac{L_{\sigma,z}}{1+L_{\sigma,z}}\vee(L_{\sigma,z}\cdot L_{\xi,x})\right)>0. \tag{2.18}$$

Also, remember $\beta_t=1\vee((1+L_{\sigma,z})(L_{\xi,x}+L(T-t)))$ such that we have for all $t\in[t',T]$:

$$\left\|\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X^k,\frac{\mathrm{d}}{\mathrm{d}\lambda}Y^k,\frac{\mathrm{d}}{\mathrm{d}\lambda}Z^k\right)\right\|_s\leq\beta_{t'}\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}+\gamma_{t'}\left\|\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X^{k-1},\frac{\mathrm{d}}{\mathrm{d}\lambda}Y^{k-1},\frac{\mathrm{d}}{\mathrm{d}\lambda}Z^{k-1}\right)\right\|_s,$$

which using $\sum_{l=0}^{k}(\gamma_{t'})^l\leq\frac{1}{1-\gamma_{t'}}$ already implies

$$\sup_{k\in\mathbb{N}_0}\left\|\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X^k,\frac{\mathrm{d}}{\mathrm{d}\lambda}Y^k,\frac{\mathrm{d}}{\mathrm{d}\lambda}Z^k\right)\right\|_s\leq\frac{\beta_{t'}\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}}{1-\gamma_{t'}}\leq K\cdot\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}, \tag{2.19}$$

where

$$K:=2\cdot\frac{1\vee((1+L_{\sigma,z})(L_{\xi,x}+LT))}{1-\frac{L_{\sigma,z}}{1+L_{\sigma,z}}\vee(L_{\sigma,z}\cdot L_{\xi,x})}<\infty. \tag{2.20}$$

Knowing that $\left(X^k,Y^k,Z^k\right)$ converges to (X,Y,Z) for fixed λ this implies by Lemma 2.1.5 that we can assume without loss of generality that the weak derivative $\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X,\frac{\mathrm{d}}{\mathrm{d}\lambda}Y,\frac{\mathrm{d}}{\mathrm{d}\lambda}Z\right)$ exists and satisfies

$$\left\|\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X,\frac{\mathrm{d}}{\mathrm{d}\lambda}Y,\frac{\mathrm{d}}{\mathrm{d}\lambda}Z\right)\right\|_s\leq CK\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}, \tag{2.21}$$

for all $t\in[t',T]$, where $C<\infty$ is some universal constant (not depending on t or X_t or t' or $\hat{t}$ or even μ,σ,f,ξ):
Here Lemma 2.1.5 is applied to each component of X,Y and Z separately and in the case of X and Y even for each fixed time $s\in[t,T]$ separately. This already explains the constant C, which is not of any concern however. Furthermore, in order to apply Lemma 2.1.5, which is formulated for integrals and not for conditional expectations, we need to decompose Ω into $\Omega_1\times\Omega_2$ such that the first component in $\omega=(\omega_1,\omega_2)$ represents all the information until time $\hat{t}$ and the second the remaining information. Then we can fix ω_1 and write for instance $\mathbb{E}_{\hat{t}}[Y_s](\omega_1,\omega_2)=\mathbb{E}[Y_s|\mathcal{F}_{\hat{t}}](\omega_1,\omega_2)=\mathbb{E}[Y_s(\omega_1,\cdot)]$,

etc. So, Lemma 2.1.5 can be applied for each fixed ω_1 separately. Also, note that norms of the form $\sup_{v\in S^{n-1}}\sqrt{\mathbb{E}[|\cdot|_v^2]}$ are equivalent to norms $\sqrt{\mathbb{E}[|\cdot|_F^2]}$, where $|\cdot|_F$ is some norm on a linear space of real-valued matrices, e.g. the Frobenius norm. This is also compensated by the constant C. Lemma 2.1.5 also allows us to assume without loss of generality that X, Y, Z are measurable functions of (λ, s, ω) and, therefore, that X_s, Y_s are also measurable functions of (λ, ω) for fixed $s \in [t, T]$.

Moreover, we would like to deduce a more restrictive bound specifically for $\frac{\mathrm{d}}{\mathrm{d}\lambda}Y$, which we will need later. Let us write $\tilde{K} := \|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\|_{\hat{t},\infty}\cdot K$ for short. Let $v\in S^{n-1}$ and $s\in[t,T]$, where $t\in[t',T]$ and $t'\in[0,T]$ is chosen according to (2.18).

Firstly, we claim that

$$\mathbb{E}_{\hat{t}}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_s^k\right|_v^2\right]\leq$$
$$\leq\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}^2+\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}C_3\tilde{K}\sqrt{T-t}+C_3\tilde{K}^2\sqrt{T-t}+L_{\sigma,z}^2\mathbb{E}_{\hat{t}}\left[\int_t^T\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}\right|_v^2\mathrm{d}r\right],\quad(2.22)$$

a.s. for all $s\in[t,T]$, where $C_3\in[0,\infty)$ is some constant depending on L and T only and in a continuous way:

This can be seen by taking to the squares both sides of (2.14), writing the right hand side as a sum of products, taking expectations and using Cauchy-Schwarz' inequality together with (2.19) several times. We will consider only a few of the products appearing as summands in this sum, the rest is analogous:

$$\mathbb{E}_{\hat{t}}\left[\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X_tv\right)^\top\int_t^s\Delta_x^{(\dots)}\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^{k-1}v\,\mathrm{d}r\right]\leq\sqrt{\mathbb{E}_{\hat{t}}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_tv\right|^2\right]\mathbb{E}_{\hat{t}}\left[\left|\int_t^s\Delta_x^{(\dots)}\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^{k-1}v\,\mathrm{d}r\right|^2\right]}\leq$$
$$\leq\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}\sqrt{\mathbb{E}_{\hat{t}}\left[(s-t)\int_t^s\left|\Delta_x^{(\dots)}\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^{k-1}v\right|^2\mathrm{d}r\right]}\leq$$
$$\leq\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}\sqrt{(s-t)\int_t^s L\sup_{w\in[t,T]}\mathbb{E}_{\hat{t},\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_w^{k-1}v\right|^2\right]\mathrm{d}r}\leq\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}L\tilde{K}(T-t).$$

In a similar fashion

$$\mathbb{E}_{\hat{t}}\left[\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X_tv\right)^\top\int_t^s\Delta_z^{(\dots)}\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v\,\mathrm{d}r\right]\leq\sqrt{\mathbb{E}_{\hat{t}}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_tv\right|^2\right]\mathbb{E}_{\hat{t}}\left[\left|\int_t^s\Delta_z^{(\dots)}\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v\,\mathrm{d}r\right|^2\right]}\leq$$
$$\leq\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}\sqrt{\mathbb{E}_{\hat{t}}\left[(s-t)\int_t^s\left|\Delta_z^{(\dots)}\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v\right|^2\mathrm{d}r\right]}\leq$$
$$\leq\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}\sqrt{\mathbb{E}_{\hat{t}}\left[(T-t)L\int_t^T\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v\right|^2\mathrm{d}r\right]}\leq\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}L\frac{1}{1+L_{\sigma,z}}\tilde{K}\sqrt{T-t}.$$

For the next product we use Itô isometry and some of the previous estimates:

$$\mathbb{E}_{\hat{t}}\left[\left(\int_t^s \Delta_x^{(\dots)}\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^{k-1}v\,\mathrm{d}r\right)^\top\int_t^s \Delta_x^{(\dots)}\sigma\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^{k-1}v\,\mathrm{d}W_r\right]\leq$$

$$\leq\sqrt{\mathbb{E}_{\hat{t}}\left[\left|\int_t^s \Delta_x^{(\dots)}\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^{k-1}v\,\mathrm{d}r\right|^2\right]\mathbb{E}_{\hat{t}}\left[\left|\int_t^s \Delta_x^{(\dots)}\sigma\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^{k-1}v\,\mathrm{d}W_r\right|^2\right]}\leq$$

$$\leq L\tilde{K}(T-t)\sqrt{\mathbb{E}_{\hat{t}}\left[\int_t^s\left|\Delta_x^{(\dots)}\sigma\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^{k-1}v\right|^2\mathrm{d}r\right]}\leq L\tilde{K}(T-t)\sqrt{\int_t^s L\sup_{w\in[t,T]}\mathbb{E}_{\hat{t},\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_w^{k-1}v\right|^2\right]\mathrm{d}r},$$

which is bounded by $L\tilde{K}(T-t)L\tilde{K}\sqrt{T-t}=L^2(T-t)^{\frac{3}{2}}\tilde{K}^2$. Similarly

$$\mathbb{E}_{\hat{t}}\left[\left(\int_t^s \Delta_z^{(\dots)}\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v\,\mathrm{d}r\right)^\top\int_t^s \Delta_x^{(\dots)}\sigma\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^{k-1}v\,\mathrm{d}W_r\right]\leq$$

$$\leq\sqrt{\mathbb{E}_{\hat{t}}\left[\left|\int_t^s \Delta_z^{(\dots)}\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v\,\mathrm{d}r\right|^2\right]\mathbb{E}_{\hat{t}}\left[\left|\int_t^s \Delta_x^{(\dots)}\sigma\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^{k-1}v\,\mathrm{d}W_r\right|^2\right]}\leq$$

$$\leq L\frac{1}{1+L_{\sigma,z}}\tilde{K}\sqrt{T-t}\sqrt{\mathbb{E}_{\hat{t}}\left[\int_t^s\left|\Delta_x^{(\dots)}\sigma\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^{k-1}v\right|^2\mathrm{d}r\right]}\leq L\frac{1}{1+L_{\sigma,z}}\tilde{K}\sqrt{T-t}L\tilde{K}\sqrt{T-t}.$$

Some of the products simply vanish due to $\hat{t}\leq t$:

$$\mathbb{E}_{\hat{t}}\left[\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X_tv\right)^\top\int_t^s \Delta_z^{(\dots)}\sigma\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v\,\mathrm{d}W_r\right]=$$

$$=\mathbb{E}_{\hat{t}}\left[\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X_tv\right)^\top\mathbb{E}_t\left[\int_t^s \Delta_z^{(\dots)}\sigma\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v\,\mathrm{d}W_r\right]\right]=\mathbb{E}_{\hat{t}}\left[\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X_tv\right)^\top 0\right]=0.$$

Finally we mention

$$\mathbb{E}_{\hat{t}}\left[\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X_tv\right)^\top\frac{\mathrm{d}}{\mathrm{d}\lambda}X_tv\right]=\mathbb{E}_{\hat{t}}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right|_v^2\right]\leq\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}$$

for a.a $\lambda\in\mathbb{R}^n,\omega\in\Omega$ together with

$$\mathbb{E}_{\hat{t}}\left[\left(\int_t^s \Delta_z^{(\dots)}\sigma\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v\,\mathrm{d}W_r\right)^\top\int_t^s \Delta_z^{(\dots)}\sigma\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v\,\mathrm{d}W_r\right]=\mathbb{E}_{\hat{t}}\left[\int_t^s\left|\Delta_z^{(\dots)}\sigma\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v\right|^2\mathrm{d}r\right]\leq$$

$$\leq L_{\sigma,z}^2\mathbb{E}_{\hat{t}}\left[\int_t^s\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}v\right|^2\mathrm{d}r\right]=L_{\sigma,z}^2\mathbb{E}_{\hat{t}}\left[\int_t^T\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}\right|_v^2\mathrm{d}r\right],$$

due to Itô isometry. ✓

At the same time using (2.17), Minkowski inequality and (2.19) one can easily deduce

$$\left(\mathbb{E}_{\hat{t}}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s^k\right|_v^2\right]+\mathbb{E}_{\hat{t}}\left[\int_s^T\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^k\right|_v^2\mathrm{d}r\right]\right)^{\frac{1}{2}}=\left(\mathbb{E}_{\hat{t}}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s^kv+\int_s^T\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^kv\,\mathrm{d}W_r\right|^2\right]\right)^{\frac{1}{2}}\leq$$

$$\leq L_{\xi,x}\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t}}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^k\right|_v^2\right]}+C_1\tilde{K}\sqrt{T-t},$$

where C_1 is some constant depending on L and T in a continuous way. Taking both sides to the squares and using (2.19) once again, we have

$$\mathbb{E}_{\hat{t}}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s^k\right|_v^2\right]+\mathbb{E}_{\hat{t}}\left[\int_s^T\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^k\right|_v^2\,\mathrm{d}r\right]\leq$$

$$\leq L_{\xi,x}^2\sup_{r\in[t,T]}\mathbb{E}_{\hat{t}}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^k\right|_v^2\right]+2L_{\xi,x}\sup_{r\in[t,T]}\sqrt{\mathbb{E}_{\hat{t}}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^k\right|_v^2\right]}C_1\tilde{K}\sqrt{T-t}+C_1^2\tilde{K}^2(T-t)\leq$$

$$\leq L_{\xi,x}^2\sup_{r\in[t,T]}\mathbb{E}_{\hat{t}}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_r^k\right|_v^2\right]+C_2\tilde{K}^2\sqrt{T-t},\quad(2.23)$$

where $C_2\in[0,\infty)$ is some constant depending on L, $L_{\xi,x}$ and T in a continuous way.

Now, by plugging (2.22) into (2.23) and regrouping the terms we have

$$\mathbb{E}_{\hat{t}}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s^k\right|_v^2\right]+\mathbb{E}_{\hat{t}}\left[\int_s^T\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^k\right|_v^2\,\mathrm{d}r\right]\leq$$

$$\leq L_{\xi,x}^2\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}^2+(L_{\xi,x}L_{\sigma,z})^2\mathbb{E}_{\hat{t}}\left[\int_t^T\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k-1}\right|_v^2\,\mathrm{d}r\right]+C_4\tilde{K}\left(\tilde{K}+\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}\right)\sqrt{T-t},\quad(2.24)$$

a.s. where C_4 is again some constant depending on L, $L_{\xi,x}$ and T in a continuous way.

Remember that according to Lemma 2.1.5 we can choose a subsequence $\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s^{k_l}(\omega_1,\cdot),\frac{\mathrm{d}}{\mathrm{d}\lambda}Z^{k_l}(\cdot,\omega_1,\cdot)\right)_{l\in\mathbb{N}}$ of $\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s^{k}(\omega_1,\cdot),\frac{\mathrm{d}}{\mathrm{d}\lambda}Z^{k}(\cdot,\omega_1,\cdot)\right)_{k\in\mathbb{N}}$ which converges to $\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s(\omega_1,\cdot),\frac{\mathrm{d}}{\mathrm{d}\lambda}Z(\cdot,\omega_1,\cdot)\right)$ in some weak $\mathcal{L}^2$ -sense. The sequence does depend on $\omega_1\in\Omega_1$, which is not a problem however: ω_1 is held fixed and the argument we are conducting works for a.a. such ω_1.
Also, due to the uniform boundedness of $\mathbb{E}_{\hat{t},\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s^k\right|_v^2\right]$ and $\mathbb{E}_{\hat{t},\infty}\left[\int_t^T\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^k\right|_v^2\,\mathrm{d}r\right]$, $k\in\mathbb{N}$, we can even assume without loss of generality that the sequences

$$\left(\mathbb{E}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s^{k_l}(\omega_1,\cdot)\right|_v^2\right]\right)_{l\in\mathbb{N}}\quad\text{and}\quad\left(\mathbb{E}\left[\int_t^T\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k_l}(\omega_1,\cdot)\right|_v^2\,\mathrm{d}r\right]\right)_{l\in\mathbb{N}}$$

converge in $\mathbb{R}$ to the values

$$\limsup_{k\to\infty}\mathbb{E}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s^{k}(\omega_1,\cdot)\right|_v^2\right]\quad\text{and}\quad\limsup_{k\to\infty}\mathbb{E}\left[\int_t^T\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k}(\omega_1,\cdot)\right|_v^2\,\mathrm{d}r\right]$$

respectively: First choose a subsequence, which satisfies this convergence of norms and then apply Lemma 2.1.5 to it to choose a subsequence of this subsequence, such that weak convergence holds as well, while convergence of norms is obviously preserved. Note, however, that these limits of norms do not necessarily have to coincide with the norms of $\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s$ and $\frac{\mathrm{d}}{\mathrm{d}\lambda}Z$ respectively.
Now, (2.24) implies

$$\lim_{l\to\infty}\mathbb{E}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s^{k_l}(\omega_1,\cdot)\right|_v^2\right]+\lim_{l\to\infty}\mathbb{E}\left[\int_s^T\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k_l}(\omega_1,\cdot)\right|_v^2\,\mathrm{d}r\right]=$$

$$=\limsup_{k\to\infty}\mathbb{E}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s^{k}(\omega_1,\cdot)\right|_v^2\right]+\limsup_{k\to\infty}\mathbb{E}\left[\int_s^T\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k}(\omega_1,\cdot)\right|_v^2\,\mathrm{d}r\right]\leq L_{\xi,x}^2\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}^2+$$

$$+(L_{\xi,x}L_{\sigma,z})^2\limsup_{k\to\infty}\mathbb{E}\left[\int_t^T\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_r^{k}(\omega_1,\cdot)\right|_v^2\,\mathrm{d}r\right]+C_4\tilde{K}\left(\tilde{K}+\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}\right)\sqrt{T-t}.$$

Because of $L_{\xi,x}L_{\sigma,z} < 1$ and the choice of (k_l) we obtain

$$\lim_{l\to\infty} \mathbb{E}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s^{k_l}(\omega_1,\cdot)\right|_v^2\right] \leq L_{\xi,x}^2 \left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}^2 + C_4\tilde{K}\left(\tilde{K} + \left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}\right)\sqrt{T-t}.$$

Using $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for all $a, b \geq 0$ we also have

$$\lim_{l\to\infty} \mathbb{E}\sqrt{\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s^{k_l}(\omega_1,\cdot)\right|_v^2\right]} \leq L_{\xi,x} \left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty} + \sqrt{C_4\tilde{K}\left(\tilde{K} + \left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}\right)\sqrt{T-t}}.$$

However, since weak convergence already implies

$$\frac{1}{|\mathcal{K}|}\int_{\mathcal{K}} \mathbb{E}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s(\omega_1,\cdot)\right|_v^2\right]\mathrm{d}\lambda \leq \limsup_{l\to\infty} \frac{1}{|\mathcal{K}|}\int_{\mathcal{K}} \mathbb{E}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s^{k_l}(\omega_1,\cdot)\right|_v^2\right]\mathrm{d}\lambda$$

for compact $\mathcal{K} \subset \mathbb{R}^n$ with $|\mathcal{K}| > 0$ we end up with the desired estimate:

$$\begin{aligned}\left(\mathbb{E}_{\hat{t}}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_s\right|_v^2\right]\right)^{\frac{1}{2}} &\leq L_{\xi,x}\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty} + \sqrt{C_4\tilde{K}\left(\tilde{K} + \left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty}\right)\sqrt{T-t}} = \\ &= \left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{\hat{t},\infty} \cdot \left(L_{\xi,x} + \sqrt{C_4K\,(K+1)\sqrt{T-t}}\right), \qquad (2.25)\end{aligned}$$

a.s. and for a.a. $\lambda \in \mathbb{R}^n$, for every $s \in [t,T] \subseteq [t',T]$. ✓

Let us now construct the decoupling field:

Firstly, let us define $t'' \in [0,T)$ as the smallest time such that

- $t' \leq t''$, where $t' \in [0,T]$ is defined at (2.18) and

$$L_{\xi,x}L_{\sigma,z} + \sqrt{C_4L_{\sigma,z}^2K\,(K+1)\sqrt{T-t''}} \leq \frac{1}{2}(L_{\xi,x}L_{\sigma,z}+1) < 1. \qquad (2.26)$$

The second requirement makes sense due to $L_{\xi,x} < L_{\sigma,z}^{-1}$. It can be equivalently reformulated as

$$C_5 := L_{\xi,x} + \sqrt{C_4K\,(K+1)\sqrt{T-t''}} \leq \frac{1}{2}(L_{\xi,x} + L_{\sigma,z}^{-1}) < L_{\sigma,z}^{-1}$$

if $L_{\sigma,z} > 0$. If $L_{\sigma,z} = 0$, we still have $C_5 < L_{\sigma,z}^{-1} = \infty$.

We now construct a mapping $u : [t'',T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ by the following procedure: For any $\lambda \in \mathbb{R}^n$ and any $t \in [t'',T]$ set

$$u(t,\cdot,\lambda) := Y_t(\lambda,\cdot),$$

where (X,Y,Z) is the unique $\mathbb{G}_t$ - solution to the FBSDE considered above with initial condition X_t given by $X_t(\lambda,\omega) := \lambda$. Note:

- $\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t = \mathrm{Id} \in \mathbb{R}^{n\times n}$, so (2.5) is satisfied with $\hat{t} := t$. Also, $\mathbb{E}_{t,\infty}\left[|X_t(\lambda,\cdot)|^2\right] = \lambda^2 < \infty$.
- $u(t,\cdot,\lambda)$ is clearly $\mathcal{F}_t$ - measurable.

- Note that $u(t,\cdot,\lambda)$ is defined for fixed $t \in [t'',T], \lambda \in \mathbb{R}^n$ only up to a null set from $\mathcal{F}$. We now claim that for every $t \in [t'',T]$ the function $u(t,\cdot,\cdot)$ can be assumed to be measurable and also Lipschitz continuous in the last component:

  For $\lambda, \tilde{\lambda} \in \mathbb{R}^n$ let $X_t(\lambda,\omega) := \lambda$ and $\tilde{X}_t(\lambda,\omega) := \tilde{\lambda}$. The associated triples (X,Y,Z) and $(\tilde{X},\tilde{Y},\tilde{Z})$ are fixed points of F and $\tilde{F}$ respectively. Therefore, using (2.11) we have

$$\left\|(X-\tilde{X},Y-\tilde{Y},Z-\tilde{Z})\right\|_w \leq \beta_t|\lambda-\tilde{\lambda}| + \gamma_t\left\|(X-\tilde{X},Y-\tilde{Y},Z-\tilde{Z})\right\|_w$$

  and, therefore,

$$(1+L_{\sigma,z})\sqrt{\mathbb{E}_{t,\infty}\left[|Y_t(\lambda,\cdot)-Y_t(\lambda,\cdot)|^2\right]} \leq \left\|(X-\tilde{X},Y-\tilde{Y},Z-\tilde{Z})\right\|_w \leq \frac{\beta_t}{1-\gamma_t}|\lambda-\tilde{\lambda}|.$$

  This implies that for almost all $\omega \in \Omega$:

$$\left|u(t,\omega,\lambda)-u(t,\omega,\tilde{\lambda})\right| \leq \frac{\beta_t}{(1-\gamma_t)(1+L_{\sigma,z})}|\lambda-\tilde{\lambda}| \qquad \forall\tilde{\lambda},\lambda \in \mathbb{Q}^n.$$

  For such ω let $\check{u}(t,\omega,\cdot)$ be the unique continuous function on $\mathbb{R}^n$ with $\check{u}(t,\omega,\lambda) = u(t,\omega,\lambda)$ for all $\lambda \in \mathbb{Q}^n$. For the remaining ω set $\check{u}$ to zero. Obviously $\check{u}(t,\cdot,\cdot)$ is measurable and

$$\left|\check{u}(t,\omega,\lambda)-\check{u}(t,\omega,\tilde{\lambda})\right| \leq \frac{\beta_t}{(1-\gamma_t)(1+L_{\sigma,z})}|\lambda-\tilde{\lambda}| \qquad \forall\tilde{\lambda},\lambda \in \mathbb{R}^n.$$

  It remains to show that for every $\lambda \in \mathbb{R}^n$ the random variables $u(t,\cdot,\lambda)$ and $\check{u}(t,\cdot,\lambda)$ coincide up to a null set: Let (λ_n) be a sequence in $\mathbb{Q}^n$ converging to λ. Using triangle inequality:

$$\begin{aligned}&\|u(t,\cdot,\lambda)-\check{u}(t,\cdot,\lambda)\|_\infty \leq \\ &\quad\leq \|u(t,\cdot,\lambda)-u(t,\cdot,\lambda_n)\|_\infty + \|u(t,\cdot,\lambda_n)-\check{u}(t,\cdot,\lambda_n)\|_\infty + \|\check{u}(t,\cdot,\lambda_n)-\check{u}(t,\cdot,\lambda)\|_\infty \leq \\ &\qquad\qquad\leq \frac{\beta_t}{(1-\gamma_t)(1+L_{\sigma,z})}|\lambda-\lambda_n| + \frac{\beta_t}{(1-\gamma_t)(1+L_{\sigma,z})}|\lambda_n-\lambda| \longrightarrow 0,\end{aligned}$$

  for $n \to \infty$. So, $\|u(t,\cdot,\lambda)-\check{u}(t,\cdot,\lambda)\|_\infty = 0$ for every $\lambda \in \mathbb{R}^n$.

- $\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{t,\infty} = \operatorname{ess\,sup}_{\lambda\in\mathbb{R}^n}\sup_{v\in S^{n-1}}\sqrt{\mathbb{E}_{t,\infty}\left[|\mathrm{Id}_{\mathbb{R}^n}|_v^2\right]} = \sup_{v\in S^{n-1}}\sqrt{|v|^2} = 1$.

- Using Lemma 2.1.4, the inequality (2.25), the property $\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_t\right\|_{t,\infty} = 1$ and the choice of t'' we obtain

$$L_{u(t,\cdot),x} = \sup_{v\in S^{n-1}}\operatorname{ess\,sup}_{\lambda\in\mathbb{R}^n,\omega\in\Omega}\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_t(\lambda,\omega)\right|_v \leq C_5 < L_{\sigma,z}^{-1},$$

  for all $t \in [t'',T]$. We also used the continuity of u in the last component.

- $\mathcal{F}_t$ - measurability of $u(t,\cdot,0)$ together with $L_{u(t,\cdot,\cdot),x} < \infty$ implies that $u(t,\cdot,\cdot) : \Omega\times\mathbb{R}^n \to \mathbb{R}^m$ is $\mathcal{F}_t\otimes\mathcal{B}(\mathbb{R}^n)$ - measurable for all $t \in [t'',T]$.

- For $t = T$ we get for every $\lambda \in \mathbb{R}^n$: $u(T,\cdot,\lambda) = \xi(X_T) = \xi(\lambda)$ a.s., so $u(T,\cdot,\cdot) = \xi$ a.e. and the terminal condition $u(T,\omega,\cdot) = \xi(\omega,\cdot)$ for a.a. $\omega \in \Omega$ is satisfied due to continuity of $u(T,\cdot,\cdot)$ and ξ in the last component.

- $\|u(t,\cdot,0)\|_\infty \leq \|(X,Y,Z)\|_w < \infty$, where (X,Y,Z) is associated with $X_t = 0$, holds due to (2.13). Remember that this value can be controlled independently of t, so $\sup_{t\in[t'',T]}\|u(t,\cdot,0)\|_\infty < \infty$.

- We claim that for any $t \in [t'', T]$ and any $\mathcal{F}_t$ - measurable initial condition $\tilde{X}_t : \Omega \to \mathbb{R}^n$ with $\mathbb{E}_{\hat{t},\infty}\left[|\tilde{X}_t|^2\right] < \infty$, where $\hat{t} \in [0, t]$, the corresponding $(\tilde{X}, \tilde{Y}, \tilde{Z}) \in \mathbb{G}_{\hat{t}}$ on $[t, T] \subseteq [t'', T]$ satisfies $\tilde{Y}_t = u(t, \tilde{X}_t)$ a.s.:

  Firstly, note that $\tilde{X}_t$ does not depend on λ, so it satisfies condition (2.5) automatically and $(\tilde{X}, \tilde{Y}, \tilde{Z}) \in \mathbb{G}_{\hat{t}}$ exist and do not depend on λ either, according to (2.21). We can assume without loss of generality that $\tilde{X}$ and $\tilde{Y}$ are continuous in time due to the forward and the backward equation. In addition we can assume without loss of generality that $\Omega = \Omega_1 \times \Omega_2$, where the projections π_1, π_2 on the two components are independent such that $\mathcal{F}_t = \sigma(\pi_1) \vee \mathcal{N}$ and $\sigma((W_r - W_t)_{r\in[T,t]}) = \sigma(\pi_2)$ and so $\tilde{X}_t$ can be assumed to be a function of ω_1. Now, fix the first component $\underline{\omega}_1 \in \Omega_1$, so $\tilde{X}_t(\underline{\omega}_1)$ becomes a constant and

  $$\tilde{X}(\cdot, (\underline{\omega}_1, \cdot)),\ \tilde{Y}(\cdot, (\underline{\omega}_1, \cdot)),\ \tilde{Z}(\cdot, (\underline{\omega}_1, \cdot))$$

  only depend on time and the second component ω_2 and solve a Lipschitz FBSDE on $[t, T]$ given by $\tilde{\mu}, \tilde{\sigma}, \tilde{f}, \tilde{\xi}$, where $\tilde{\mu}(s, \omega_2, x, y, z) := \mu(s, (\underline{\omega}_1, \omega_2), x, y, z)$, $\tilde{\xi}(\omega_2, x) := \xi((\underline{\omega}_1, \omega_2), x)$ etc. So, $\tilde{\mu}, \tilde{\sigma}, \tilde{f}, \tilde{\xi}$ depend on ω only through the second component and are still progressively measurable. They also inherit Lipschitz-continuity properties from μ, σ, f, ξ for almost every $\underline{\omega}_1 \in \Omega_1$.

  On the other hand for X_t given by $X_t(\lambda, \omega) := \lambda \in \mathbb{R}^n$ we can consider (X, Y, Z) associated with this initial condition and the parameters μ, σ, f, ξ such that $(X, Y, Z) \in \mathbb{G}_t$ for every $\lambda \in \mathbb{R}^n$. X, Y, Z are functions of λ, time and ω. We again assume that X, Y are continuous in time. Now, for $\underline{\omega}_1$ fixed above choose $\lambda := \tilde{X}_t(\underline{\omega}_1)$. Consider the FBSDE satisfied by $X(\lambda, \cdot, \cdot), Y(\lambda, \cdot, \cdot), Z(\lambda, \cdot, \cdot)$ and replace $\omega = (\omega_1, \omega_2) \in \Omega$ with $(\underline{\omega}_1, \omega_2)$, such that

  $$X(\lambda, \cdot, (\underline{\omega}_1, \cdot)),\ Y(\lambda, \cdot, (\underline{\omega}_1, \cdot)),\ Z(\lambda, \cdot, (\underline{\omega}_1, \cdot))$$

  solve the same FBSDE as

  $$\tilde{X}(\cdot, (\underline{\omega}_1, \cdot)),\ \tilde{Y}(\cdot, (\underline{\omega}_1, \cdot)),\ \tilde{Z}(\cdot, (\underline{\omega}_1, \cdot))$$

  and must, therefore, coincide because they are both in $\mathbb{G}_{\hat{t}}$ for almost all $\underline{\omega}_1 \in \Omega$.
  This shows $\tilde{Y}_t(\underline{\omega}_1, \cdot) = Y(\lambda, t, (\underline{\omega}_1, \cdot)) = u(t, (\underline{\omega}_1, \cdot), \lambda) = u(t, (\underline{\omega}_1, \cdot), \tilde{X}_t(\underline{\omega}_1))$ a.s. for almost every $\underline{\omega}_1 \in \Omega_1$. ✓

- More generally: For any $\mathcal{F}_t$ - measurable initial condition $\tilde{X}_t : \Omega \to \mathbb{R}^n$ with $\mathbb{E}_{\hat{t},\infty}\left[|\tilde{X}_t|^2\right] < \infty$, where $\hat{t} \in [0, t]$, the corresponding $(\tilde{X}, \tilde{Y}, \tilde{Z}) \in \mathbb{G}_{\hat{t}}$ on $[t, T] \subseteq [t'', T]$ satisfies $\tilde{Y}_s = u(t, \tilde{X}_s)$ a.s. for all $s \in [t, T]$:

  Since $(\tilde{X}, \tilde{Y}, \tilde{Z}) \in \mathbb{G}_{\hat{t}}$ we have $\mathbb{E}_{\hat{t},\infty}\left[|\tilde{X}_s|^2\right] < \infty$ for all $s \in [t, T]$. Viewing $\tilde{X}_s$ as an initial condition for the FBSDE on the interval $[s, T]$ the previous statement provides $\tilde{Y}_s = u(t, \tilde{X}_s)$ a.s. ✓

- Let $t \in [t'', T]$. As above we can assume $\Omega = \Omega_1 \times \Omega_2$ with $\mathcal{F}_t = \sigma(\pi_1) \vee \mathcal{N}$ and $\sigma((W_r - W_t)_{r\in[T,t]}) = \sigma(\pi_2)$. Fix $\underline{\omega}_1 \in \Omega_1$ and remember $\tilde{\mu}, \tilde{\sigma}, \tilde{f}, \tilde{\xi}$ defined above. We fix the following notation for later use: Let $\tilde{X}^{(t,\underline{\omega}_1)}, \tilde{Y}^{(t,\underline{\omega}_1)}, \tilde{Z}^{(t,\underline{\omega}_1)}$ be the $\mathbb{G}_t$ - processes on $\mathbb{R}^n \times [t, T] \times \Omega_2$ associated with the initial condition $X_t(\lambda, \omega_2) := \lambda$, where $(\lambda, \omega_2) \in \mathbb{R}^n \times \Omega_2$, such that $\tilde{X}^{(t,\underline{\omega}_1)}, \tilde{Y}^{(t,\underline{\omega}_1)}$ are continuous in time. For every $\lambda \in \mathbb{R}^n$ and $s \in [t, T]$ we have $\tilde{Y}^{(t,\underline{\omega}_1)}_s(\lambda, \omega_2) = u(s, (\underline{\omega}_1, \omega_2), \tilde{X}^{(t,\underline{\omega}_1)}_s(\lambda, \omega_2))$ for a.a. $\omega_2 \in \Omega_2$ and a.a. $\underline{\omega}_1 \in \Omega_1$ as we saw above.

We can now assume without loss of generality that u is progressively measurable and in some sense right-continuous (Lemma 2.1.3).

Let us now show that $u : [t'', T] \times \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is indeed a decoupling field:

Choose any $t_1 < t_2$ from $[t'', T]$ and *any* $\mathcal{F}_{t_1}$ - measurable initial condition X_{t_1}. Assume without loss of generality that X_{t_1} is a function of $\underline{\omega}_1 \in \Omega$ only and define for $\omega = (\underline{\omega}_1, \omega_2) \in \Omega$ and $s \in [t_1, t_2]$

$$X_s(\omega) := \tilde{X}^{(t_1,\underline{\omega}_1)}(X_{t_1}(\underline{\omega}_1), s, \omega_2),\ Y_s(\omega) := \tilde{Y}^{(t_1,\underline{\omega}_1)}(X_{t_1}(\underline{\omega}_1), s, \omega_2),\ Z_s(\omega) := \tilde{Z}^{(t_1,\underline{\omega}_1)}(X_{t_1}(\underline{\omega}_1), s, \omega_2).$$

It is straightforward to check that X, Y, Z are progressively measurable, satisfy the forward equation, the backward equation and the decoupling condition: All these properties are inherited from $\tilde{X}^{(t_1,\underline{\omega}_1)}, \tilde{Y}^{(t_1,\underline{\omega}_1)}, \tilde{Z}^{(t_1,\underline{\omega}_1)}$, which satisfy the three equations a.s. for every s and λ, so they are fulfilled for almost all $\omega_2 \in \Omega_2$ for $\lambda := X_{t_1}(\underline{\omega}_1)$ and a fixed $s \in [t_1, t_2]$ and so they hold for almost all $\omega \in \Omega$, since the argument works for a.a. $\underline{\omega}_1 \in \Omega$. It is also sufficient to consider only countably many $s \in [t_1, t_2] \cap \mathbb{Q} \cup \{t_1, t_2\}$ due to continuity of $\tilde{X}^{(t,\underline{\omega}_1)}, \tilde{Y}^{(t,\underline{\omega}_1)}$ and right-continuity of u from Lemma 2.1.3.
The initial condition is satisfied via definition of X and $\tilde{X}^{(t_1,\underline{\omega}_1)}$. The terminal condition $u(T, \omega, \cdot) = \xi(\omega, \cdot)$ has already been discussed. ✓

We can also show uniqueness of processes X, Y, Z on $[t, t_2] \times \Omega \times \mathbb{R}^n$ solving the forward and backward equations together with the decoupling condition via u and some initial condition X_t, where $[t, t_2] \subseteq [t'', T]$ and X_t is $\mathcal{F}_t$ - measurable and satisfies $\mathbb{E}_{\hat{t},\infty}[|X_t|^2] < \infty$ for some $\hat{t} \in [0, t]$:

The triples (X, Y, Z) for deterministic initial conditions constructed so far are obviously in $\mathbb{G}_{\hat{t}}$. Assume that there is another triple $(\hat{X}, \hat{Y}, \hat{Z})$ with the required three properties. If we can show that this triple is also in $\mathbb{G}_{\hat{t}}$, we are done, due to uniqueness of $\mathbb{G}_{\hat{t}}$ - solutions to FBSDEs.

Let $\tau \in [t, t_2]$ be any stopping time with a countable range. It is straightforward to verify that the triple

$$(\hat{X}_{\cdot\wedge\tau}, \hat{Y}_{\cdot\wedge\tau}, \hat{Z}\mathbf{1}_{\{\cdot\leq\tau\}})$$

solves the FBSDE on $[t, t_2]$ given by $\hat{\mu} := \mu\mathbf{1}_{\{\cdot\leq\tau\}}$, $\hat{\sigma} := \sigma\mathbf{1}_{\{\cdot\leq\tau\}}$, $\hat{f} := f\mathbf{1}_{\{\cdot\leq\tau\}}$ and terminal condition $\hat{\xi} := u(\tau, \cdot)$. We only check the backward equation for $\hat{Y}_{\cdot\wedge\tau}, \hat{Z}\mathbf{1}_{\{\cdot\leq\tau\}}$:

$$\hat{Y}_{s\wedge\tau} = \hat{Y}_{t_2\wedge\tau} - \int_{s\wedge\tau}^{t_2\wedge\tau} f(r, \hat{X}_r, \hat{Y}_r, \hat{Z}_r)\,\mathrm{d}r - \int_{s\wedge\tau}^{t_2\wedge\tau} \hat{Z}_r\,\mathrm{d}W_r =$$
$$= \hat{Y}_\tau - \int_s^{t_2} \mathbf{1}_{\{r\leq\tau\}} f(r, \hat{X}_r, \hat{Y}_r, \hat{Z}_r)\,\mathrm{d}r - \int_s^{t_2} \mathbf{1}_{\{r\leq\tau\}} \hat{Z}_r\,\mathrm{d}W_r$$

Note here that the decoupling condition can be easily generalized to stopping times such that $\hat{Y}_\tau = u(\tau, \hat{X}_\tau)$ if τ has a countable range. So, we can write

$$\hat{Y}_{s\wedge\tau} = u(\tau, \hat{X}_\tau) - \int_s^{t_2} \mathbf{1}_{\{r\leq\tau\}} f(r, \hat{X}_{r\wedge\tau}, \hat{Y}_{r\wedge\tau}, \mathbf{1}_{\{r\leq\tau\}}\hat{Z}_r)\,\mathrm{d}r - \int_s^{t_2} \mathbf{1}_{\{r\leq\tau\}} \hat{Z}_r\,\mathrm{d}W_r =$$
$$= \hat{\xi}(\hat{X}_{t_2\wedge\tau}) - \int_s^{t_2} \hat{f}(r, \hat{X}_{r\wedge\tau}, \hat{Y}_{r\wedge\tau}, \mathbf{1}_{\{r\leq\tau\}}\hat{Z}_r)\,\mathrm{d}r - \int_s^{t_2} \mathbf{1}_{\{r\leq\tau\}} \hat{Z}_r\,\mathrm{d}W_r.$$

Note that $L_{\hat{\xi},x} \leq L_{u,x} < L_{\sigma,z}^{-1} \leq L_{\hat{\sigma},z}^{-1}$ and also $\|\hat{\xi}(0)\|_\infty \leq \sup_{s\in[t'',T]} \|u(s, \cdot, 0)\|_\infty < \infty$.
This new FBSDE on $[t, t_2]$ has the same Lipschitz properties as the initial one and we can apply uniqueness of $\mathbb{G}_{\hat{t}}$ - solutions. If τ is chosen such that $(\hat{X}_{\cdot\wedge\tau}, \hat{Y}_{\cdot\wedge\tau}, \hat{Z}\mathbf{1}_{\{\cdot\leq\tau\}})$ is in $\mathbb{G}_{\hat{t}}$, we can control its $\|\cdot\|_w$-norm independently of τ, according to (2.13). Using localization we can control the $\|\cdot\|_w$-norm of $(\hat{X}, \hat{Y}, \hat{Z})$ itself, which shows that it is in $\mathbb{G}_{\hat{t}}$ and we are done. ✓

Uniqueness of u on $[t'', T]$ follows easily from our knowledge, that processes X, Y, Z associated with decoupling fields, which satisfy the two properties $L_{u,x} < L_{\sigma,z}^{-1}$ and $\sup_{s\in[t'',T]} \|u(s, \cdot, 0)\|_\infty < \infty$, are automatically in $\mathbb{G}_t$, as we have seen above, at least if $X_t = x \in \mathbb{R}^n$:

Since the FBSDE on an interval $[t,T]$ with $t \in [t',T]$ is the same for all such decoupling fields, the $\mathbb{G}_t$ - solution (X,Y,Z) is also the same for all such decoupling fields. This already uniquely determines $u(t,\cdot,x) = Y_t(x,\cdot)$. ✓
Now, the proof of Theorem 2.2.1 is complete. □

Remark 2.2.2. Consider all t large enough such that the above construction works and s.t. t has the properties required in Theorem 2.2.1 (e.g. $t = t''$). One can easily derive from the above proof that the supremum of all corresponding $h = T - t$ can be bounded away from 0 by a positive bound, which depends only on

- the Lipschitz constant L of μ, σ and f w.r.t. the last 3 components, T, $L_{\sigma,z}$,
- L_ξ and $L_\xi \cdot L_{\sigma,z} < 1$,

and which is monotonically decreasing in these values:

In order to see this remember the definitions of γ_t at (2.12) and K at (2.20) together with the choice of t' at (2.18) and t'' at (2.26).

Remark 2.2.3. As we have seen (2.25) implies that our decoupling field u on $[t,T]$ satisfies

$$L_{u(s,\cdot),x} \leq L_{\xi,x} + C(T-s)^{\frac{1}{4}},$$

where C is some constant which does not depend on $s \in [t,T]$:

This is because $L_{u(t_1,\cdot),x} = \sup_{v \in S^{n-1}} \operatorname{ess\,sup}_{\lambda \in \mathbb{R}^n, \omega \in \Omega} \left| \frac{\mathrm{d}}{\mathrm{d}\lambda} Y_{t_1}(\lambda,\omega) \right|_v$ for $X_{t_1}(\lambda,\omega) := \lambda$, $t_1 \in [t,T]$.

More precisely, C depends only on T, L, $L_{\xi,x}$, $L_{\xi,x} L_{\sigma,z}$ and is monotonically increasing in these values.

Remark 2.2.4. If we do not care about decoupling fields but are only interested in processes X, Y, Z solving the forward and the backward equations together with $Y_T = \xi(X_T)$ for given $t_1 \in [t,T], t_2 := T$ and initial condition X_{t_1}, where t is sufficiently close to T as required by Remark 2.2.2, the above construction does provide existence but not uniqueness.

In order to have uniqueness we need an additional restriction, e.g. $(X,Y,Z) \in \mathbb{G}_0$, where $\mathbb{G}_0$ is defined at (2.6). Under this condition we would get not only uniqueness but also the decoupling condition $Y_s = u(s, X_s)$ a.s. as we have seen.

Conversely the two conditions $X_{t_1} = x \in \mathbb{R}^n$ and $Y_s = u(s,X_s)$, $s \in [t,T]$ would also suffice for uniqueness: In fact we saw that this already implies $(X,Y,Z) \in \mathbb{G}_{t_1} \subseteq \mathbb{G}_0$. For the argument we only needed that X, Y, Z solve the FBSDE on $[t_1, T]$ and $Y_s = u(s, X_s)$ holds true with some adapted map u s.t. $L_{u,x} < L_{\sigma,z}^{-1}$ and $\sup_{s \in [t,T]} \|u(s,\cdot,0)\|_\infty < \infty$.

2.3 Some examples

We first demonstrate that the assumption $L_{\xi,x} < L_{\sigma,z}^{-1}$ cannot be dropped or weakened:

Example 2.3.1. For instance, consider the forward backward problem

$$X_s = x + \int_t^s (\sigma_0 + Z_r)\,\mathrm{d}W_r,$$

$$Y_s = X_T - \int_s^T Z_r\,\mathrm{d}W_r, \quad s \in [t,T].$$

This means

- $n = m = d = 1$,

- μ and f vanish,
- σ is defined via $\sigma(s,x,y,z) = \sigma_0 + z$, where $\sigma_0 \in \mathbb{R}\backslash\{0\}$ is some constant,
- $\xi = \mathrm{Id}_{\mathbb{R}}$.

We obviously have $L_{\xi,x} = 1$ and $L_{\sigma,z} = 1$ and so $L_{\xi,x} = L_{\sigma,z}^{-1}$.
We now claim that this problem cannot have a progressive solution, no matter how small $T-t>0$ is chosen.
In fact, the forward equation implies

$$X_T - X_s = \int_s^T (\sigma_0 + Z_r)\,\mathrm{d}W_r = \sigma_0(W_T - W_s) + \int_s^T Z_r\,\mathrm{d}W_r$$

or

$$X_T - \int_s^T Z_r\,\mathrm{d}W_r = X_s + \sigma_0(W_T - W_s), \quad s \in [t,T].$$

Together with the backward equation we obtain

$$Y_s = X_s + \sigma_0(W_T - W_s),$$

which for $s = t$ means

$$Y_t - x = \sigma_0(W_T - W_t).$$

This cannot be true, however, since Y_t is $\mathcal{F}_t$ - measurable and $\sigma_0(W_T - W_t)$ is a non-degenerate Gaussian random variable independent of $\mathcal{F}_t$.

The requirement to choose $T-t$ small enough cannot be omitted either:

Example 2.3.2. Let $T \in [1,\infty)$. For $t \in [0,T)$ consider the following FBSDE on the interval $[t,T]$:

$$X_s = x + \int_t^s Y_r\,\mathrm{d}r,$$

$$Y_s = X_T - \int_s^T Z_r\,\mathrm{d}W_r, \quad s \in [t,T].$$

This means

- $n = m = d = 1$,
- σ and f vanish, in particular $L_{\sigma,z} = 0$ and $L_{\sigma,z}^{-1} = \infty$,
- μ is defined via $\mu(s,x,y,z) = y$,
- $\xi = \mathrm{Id}_{\mathbb{R}}$.

For $t \in (T-1,T]$ the problem has a decoupling field

$$u(s,x) = \frac{x}{1-(T-s)}, \quad s \in [t,T],$$

such that

$$X_s = x + (s-t)\frac{x}{1-(T-t)} = x\frac{1-(T-s)}{1-(T-t)},$$

$$Y_s = \frac{x}{1-(T-t)},$$

$$Z_s = 0, \quad s \in [t,T].$$

This is straightforward to verify. Note also $L_{u,x} < \infty$ and $\sup_{s\in[t,T]} \|u(s,\cdot,0)\|_\infty < \infty$. We will see later that this u is unique among all decoupling fields with these two properties (Corollary 2.5.3).
Note that $u(t,x)$ tends to ∞ for $t \downarrow (T-1)$ for all $x \neq 0$ indicating that the problem might be ill-posed for $t = T-1$. In fact we will see later that for $t = T-1$ there is no decoupling field u with $L_{u,x} < \infty$ and $\sup_{s\in[t,T]} \|u(s,\cdot,0)\|_\infty < \infty$:
Otherwise this u would have to be continuous according to Lemma 2.5.15, which is applicable due to Corollary 2.5.4 and Lemma 2.5.13, contradicting $\lim_{t\downarrow(T-1)} u(t,x) = \infty$ for $x \neq 0$.

2.4 Regularity

Definition 2.4.1. Let $u : [t,T]\times\Omega\times\mathbb{R}^n \to \mathbb{R}^m$ be a decoupling field to $(\xi,(\mu,\sigma,f))$. We call u *weakly regular*, if $L_{u,x} < L_{\sigma,z}^{-1}$ and $\sup_{s\in[t,T]} \|u(s,\cdot,0)\|_\infty < \infty$.

The decoupling field constructed in Theorem 2.2.1 is weakly regular as we have seen.
Weak regularity implies weak differentiability of u w.r.t. x (Lemma 2.1.4). It also allows to assume that u is progressively measurable (Lemma 2.1.3).
In practice, however, it is important to have explicit knowledge about the regularity of the associated (X,Y,Z). For instance, it is important to know in which spaces the processes live, and how they react to changes in the initial value. Specifically it can be very useful to have differentiability of X,Y,Z w.r.t. the initial value.

Definition 2.4.2. Let $u : [t,T]\times\Omega\times\mathbb{R}^n \to \mathbb{R}^m$ be a weakly regular decoupling field to $(\xi,(\mu,\sigma,f))$. We call u *strongly regular* if for all fixed $t_1,t_2 \in [t,T]$, $t_1 \leq t_2$, the processes X,Y,Z arising in the defining property of a decoupling field are a.e. unique for each *constant* initial value $X_{t_1} = x \in \mathbb{R}^n$ and satisfy

$$\sup_{s\in[t_1,t_2]} \mathbb{E}_{t_1,\infty}[|X_s|^2] + \sup_{s\in[t_1,t_2]} \mathbb{E}_{t_1,\infty}[|Y_s|^2] + \mathbb{E}_{t_1,\infty}\left[\int_{t_1}^{t_2} |Z_s|^2\,\mathrm{d}s\right] < \infty \quad \forall x \in \mathbb{R}^n. \tag{2.27}$$

In addition they must be measurable as functions of (x,s,ω) and even weakly differentiable w.r.t. $x \in \mathbb{R}^n$ such that for every $s \in [t_1,t_2]$ the mappings X_s and Y_s are measurable functions of (x,ω) and even weakly differentiable w.r.t. x such that

$$\begin{aligned}
&\operatorname{ess\,sup}_{x\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \sup_{s\in[t_1,t_2]} \mathbb{E}_{t_1,\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}x}X_s\right|_v^2\right] < \infty,\\
&\qquad\operatorname{ess\,sup}_{x\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \sup_{s\in[t_1,t_2]} \mathbb{E}_{t_1,\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}x}Y_s\right|_v^2\right] < \infty,\\
&\qquad\qquad\operatorname{ess\,sup}_{x\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \mathbb{E}_{t_1,\infty}\left[\int_{t_1}^{t_2}\left|\frac{\mathrm{d}}{\mathrm{d}x}Z_s\right|_v^2\mathrm{d}s\right] < \infty.
\end{aligned} \tag{2.28}$$

We say that a decoupling field u on $[t,T]$ is *strongly regular* on a subinterval $[t_1,t_2] \subseteq [t,T]$ if u restricted to $[t_1,t_2]$ is a strongly regular decoupling field for $(u(t_2,\cdot),(\mu,\sigma,f))$.

Remark 2.4.3. Note that under the forward equation a.e.-uniqueness of X,Y,Z already implies a.s.-uniqueness of X_s for every fixed time $s \in [t_1,t_2]$. Using the decoupling condition $Y_s = u(s,X_s)$ a.s. this implies a.s.-uniqueness of Y_s for every fixed time $s \in [t_1,t_2]$ as well. So, the above requirement of measurability and differentiability of X_s, Y_s makes sense.
Also, observe that if X,Y,Z are measurable functions of (x,s,ω), the forward equation and measurability of μ,σ imply that X_s will be a measurable function of (x,ω) for every $s \in [t_1,t_2]$. But then

measurability of $u(s,\cdot)$ implies that Y_s will be a measurable function of (x,ω) for every $s\in[t_1,t_2]$ as well.
Finally note that if we define a mapping $(\tilde{X},\tilde{Y},\tilde{Z})$ on $\mathbb{R}^n\times[t_1,t_2]\times\Omega$ point-wise for every $x\in\mathbb{R}^n$ such that the processes $\tilde{X}(x,\cdot),\tilde{Y}(x,\cdot),\tilde{Z}(x,\cdot)$ defined on $[t_1,t_2]\times\Omega$ satisfy the forward equation, the backward equation and the decoupling condition (almost surely for every $s\in[t_1,t_2]$), the resulting $(\tilde{X},\tilde{Y},\tilde{Z})$ does not have to be measurable, of course, even though $(\tilde{X}(x,\cdot),\tilde{Y}(x,\cdot),\tilde{Z}(x,\cdot))$ is unique up to a null set. So, in the definition we implicitly require that there *exist* measurable mappings X,Y,Z on $\mathbb{R}^n\times[t_1,t_2]\times\Omega$ such that for every x the processes $X(x,\cdot),Y(x,\cdot),Z(x,\cdot)$ are the (up to a null set) unique processes satisfying the forward equation, the backward equation and the decoupling condition and such that X,Y,Z,X_s,Y_s, $s\in[t_1,t_2]$ have the required measurability and differentiability properties.

Remark 2.4.4. We can see from the proof of Theorem 2.2.1 that the decoupling field u constructed there is strongly regular:

Uniqueness of X,Y,Z follows from Remark 2.2.4, while (2.27) and (2.28) follow from (2.13) and (2.21) for $\hat{t}:=t_1$, $\lambda:=x$ and $X_{t_1}(x,\omega):=x\in\mathbb{R}^n$. ✓

Remark 2.4.5. Note that the values $L_{u,x}$ and $\sup_{s\in[t,T]}\|u(s,\cdot,0)\|_\infty$ do not change if we replace u with one of its modifications.

Lemma 2.4.6. *Let g,μ,σ,f be as in Theorem 2.2.1, let $0\le s<t<T$ and let u be a weakly regular decoupling field for $(\xi,(\mu,\sigma,f))$ on $[s,T]$.*
If u is strongly regular on $[s,t]$ and $T-t$ is small enough as required in Theorem 2.2.1 resp. Remark 2.2.2, then u is strongly regular on $[s,T]$.

Proof. We only need to investigate the case $s\le t_1\le t\le t_2\le T$. Otherwise we just have to apply the regularity of u on either $[s,t]$ or $[t,T]$.
Since u is a decoupling field there exist progressive X,Y,Z on $[t_1,T]$ satisfying the forward equation, the backward equation, the decoupling condition and the initial condition $X_{t_1}=x\in\mathbb{R}^n$. Due to strong regularity of u on $[s,t]$ these X,Y,Z are a.e. unique at least on $[t_1,t]$ and satisfy the required measurability and differentiability conditions on $[t_1,t]$ such that

$$\sup_{s\in[t_1,t]}\mathbb{E}_{t_1,\infty}[|X_s|^2]+\sup_{s\in[t_1,t]}\mathbb{E}_{t_1,\infty}[|Y_s|^2]+\mathbb{E}_{t_1,\infty}\left[\int_{t_1}^t|Z_s|^2\,\mathrm{d}s\right]<\infty\quad\forall x\in\mathbb{R}^n,$$

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^n}\sup_{v\in S^{n-1}}\sup_{r\in[t_1,t]}\mathbb{E}_{t_1,\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}x}X_r\right|_v^2\right]<\infty,$$

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^n}\sup_{v\in S^{n-1}}\sup_{r\in[t_1,t]}\mathbb{E}_{t_1,\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}x}Y_r\right|_v^2\right]<\infty,$$

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^n}\sup_{v\in S^{n-1}}\mathbb{E}_{t_1,\infty}\left[\int_{t_1}^t\left|\frac{\mathrm{d}}{\mathrm{d}x}Z_r\right|_v^2\mathrm{d}r\right]<\infty.$$

In particular, $\operatorname{ess\,sup}_{x\in\mathbb{R}^n}\sup_{v\in S^{n-1}}\mathbb{E}_{t_1,\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}x}X_t\right|_v^2\right]<\infty$ and also $\mathbb{E}_{t_1,\infty}[|X_t|^2]<\infty$ for all $x\in\mathbb{R}^n$.
Now, following the construction of Theorem 2.2.1 we obtain for this $X_t:\mathbb{R}^n\times\Omega\to\mathbb{R}^n$ progressive processes $\check{X},\check{Y},\check{Z}$ on $\mathbb{R}^n\times[t,T]\times\Omega$ which satisfy the required measurability and differentiability conditions together with

$$\sup_{s\in[t,T]}\mathbb{E}_{t_1,\infty}[|\check{X}_s|^2]+\sup_{s\in[t,T]}\mathbb{E}_{t_1,\infty}[|\check{Y}_s|^2]+\mathbb{E}_{t_1,\infty}\left[\int_t^T|\check{Z}_s|^2\,\mathrm{d}s\right]<\infty\quad\forall x\in\mathbb{R}^n,$$

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \sup_{r\in[t,T]} \mathbb{E}_{t_1,\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}x}\check{X}_r\right|_v^2\right] < \infty,$$

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \sup_{r\in[t,T]} \mathbb{E}_{t_1,\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}x}\check{Y}_r\right|_v^2\right] < \infty,$$

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \mathbb{E}_{t_1,\infty}\left[\int_t^T \left|\frac{\mathrm{d}}{\mathrm{d}x}\check{Z}_r\right|_v^2 \mathrm{d}r\right] < \infty,$$

and such that $\check{X}, \check{Y}, \check{Z}$ solve the FBSDE given by $(\xi, (\mu, \sigma, f))$ on $[t, T]$, satisfy the decoupling condition $\check{Y}_r = u(r, \check{X}_r)$ a.s., $r \in [t, T]$ and finally the initial condition $\check{X}_t = X_t$ a.s.. We have seen at the end of proof of Theorem 2.2.1 that processes with the latter three properties are already a.e.-unique.
Since these three properties are also satisfied by X, Y, Z on $[t, T]$ we have $(\check{X}, \check{Y}, \check{Z}) = (X, Y, Z)$ a.e. on $[t, T]$. So, (X, Y, Z) has the desired measurability and differentiability properties on the whole of $[t_1, T]$ and in particular on $[t_1, t_2]$.
This shows strong regularity of u. □

2.5 Extension to large intervals

In the following we employ local results from the previous sections to obtain global existence and uniqueness in a sense specified later. We will extensively use a simple basic argument which we will refer to as *small interval induction*.

Lemma 2.5.1 (Small interval induction, backward). *Let $T_1 < T_2$ be real numbers and let $S \subseteq [T_1, T_2]$ s.t.*

- *$T_2 \in S$,*
- *there exists an $h > 0$ s.t. $\forall s \in \mathbb{R}$: $s \in S \implies [s - h, s] \cap [T_1, T_2] \subseteq S$.*

Then $S = [T_1, T_2]$. In particular, $T_1 \in S$.

Proof. Let $s_{\min}$ be the infimum of all $s \in S$ such that $[s, T_2] \subseteq S$. Since $T_2 \in S$ this value is in $[T_1, T_2]$. Obviously $(s_{\min}, T_2] \subseteq S$. We claim that $s_{\min} = T_1$. Assume otherwise. Then $(s_{\min} + h/2) \wedge T_2 \in S$ implies $[T_1 \vee (s_{\min} - h/2), (s_{\min} + h/2) \wedge T_2] \subseteq S$, which in turn leads to $[T_1 \vee (s_{\min} - h/2), T_2] \subseteq S$ contradicting the definition of $s_{\min}$. Thus $s_{\min} = T_1$. In particular, $(T_1 + h) \wedge T_2 \in S$, which implies $T_1 \in S$. □

Similarly one can show:

Lemma 2.5.2 (Small interval induction, forward). *Let $T_1 < T_2$ be real numbers and let $S \subseteq [T_1, T_2]$ s.t.*

- *$T_1 \in S$,*
- *there exists an $h > 0$ s.t. $[s, s + h] \cap [T_1, T_2] \subseteq S$ for all $s \in S$.*

Then $S = [T_1, T_2]$. In particular, $T_2 \in S$.

We omit the proof due to analogy and simplicity of the statement.

Here is a first application of this technique.

Corollary 2.5.3 (Global uniqueness)**.** *Let μ, σ, f, ξ be as in Theorem 2.2.1. Assume that there are two weakly regular decoupling fields $u^{(1)}, u^{(2)}$ to the corresponding problem on some interval $[t, T]$. Then $u^{(1)} = u^{(2)}$ up to modifications.*

Proof. Let $S \subseteq [t, T]$ be the set of all times $s \in [t, T]$, s.t. $u^{(1)}(s, \omega, \cdot) = u^{(2)}(s, \omega, \cdot)$ for a.a. $\omega \in \Omega$.

- Obviously $T \in S$, due to the terminal condition.
- Let $s \in S$ be arbitrary. According to Theorem 2.2.1 there exists an $h > 0$ such that there is a unique decoupling field $\tilde{u}$ to $(u^{(1)}(s, \cdot), (\mu, \sigma, f)) = (u^{(2)}(s, \cdot), (\mu, \sigma, f))$ on the interval $[(s-h)\vee t, s]$ s.t. $L_{\tilde{u},x} < L^{-1}_{\sigma,z}$, $\sup_{r\in[(s-h)\vee t,s]} \|\tilde{u}(r, \cdot, 0)\|_\infty < \infty$, while h can be chosen independently of s according to Remark 2.2.2. The three decoupling fields $\tilde{u}, u^{(1)}, u^{(2)}$ coincide on $[(s-h) \vee t, s]$ according to Theorem 2.2.1 and, hence, $[(s-h) \vee t, s] \subseteq S$.

This shows $S = [t, T]$ by small interval induction. □

After having shown uniqueness of u we show its strong regularity. For this purpose we use the forward version of small interval induction.

Corollary 2.5.4 (Global regularity)**.** *Let μ, σ, f, ξ be as in Theorem 2.2.1. Assume that there exists a weakly regular decoupling field u to this problem on some interval $[t, T]$.Then u is strongly regular.*

Proof. Let $S \subseteq [t, T]$ be the set of all times $s \in [t, T]$ s.t. u is strongly regular on $[t, s]$.

- Obviously $t \in S$, since for the interval $[t, t]$ the a.e.-only choice for X, Y, Z is $Z(x, t, \omega) = 0$, $X(x, t, \omega) = x$ and $Y(x, t, \omega) = u(t, \omega, x)$ for $(x, t, \omega) \in \mathbb{R}^n \times [t, t] \times \Omega$. This (X, Y, Z) satisfies the required measurability and differentiability conditions together with (2.27) and (2.28) due to weak regularity of u which provides $L_{u(t,\cdot),x} < L^{-1}_{\sigma,z}$ and $\|u(t, \cdot, 0)\|_\infty < \infty$. We also used Lemma 2.1.4.
- Let $s \in S$ be arbitrary. According to Lemma 2.4.6 there exists an $h > 0$ s.t. u is strongly regular on $[t, (s+h) \wedge T]$ since $L_{u((s+h)\wedge T,\cdot),x} < L^{-1}_{\sigma,z}$. Recalling Remark 2.2.2 and the weak regularity which provides $L_{u((s+h)\wedge T,\cdot),x} \leq L_{u,x} < L^{-1}_{\sigma,z}$, we can choose h independently of s.

This shows $S = [t, T]$ by small interval induction. □

Notice that Corollary 2.5.3 only provides uniqueness of weakly regular decoupling fields, not uniqueness of processes (X, Y, Z) solving the FBSDE in the classical sense. However, we can show Corollary 2.5.5 below. Remember for this result the definition of the space $\mathbb{G}_{\hat{t}}$ at (2.6):

Corollary 2.5.5. *Let μ, σ, f, ξ be as in Theorem 2.2.1. Assume that there exists a weakly regular decoupling field u on some interval $[t, T]$.*
Then for any deterministic initial condition $X_t = x \in \mathbb{R}^n$ there is a unique $\mathbb{G}_0$ - solution X, Y, Z of the FBSDE on $[t, T]$.

Proof. The existence of (X, Y, Z) follows directly from strong regularity of u (Corollary 2.5.4). In fact we even have $(X, Y, Z) \in \mathbb{G}_t$. Let us show uniqueness. Assume we have another $\mathbb{G}_0$ - solution $(\hat{X}, \hat{Y}, \hat{Z})$. Due to strong regularity, we only need to show the decoupling condition $\hat{Y}_s = u(s, \hat{X}_s)$, $s \in [t, T]$:
Choose a $t' \in [t, T]$ close enough to T according to Remark 2.2.2. Then we have $\hat{Y}_s = u(s, \hat{X}_s)$ for $s \in [t', T]$ according to Remark 2.2.4. In particular, we have $\hat{Y}_{t'} = u(t', \hat{X}_{t'})$ a.s., which serves as a new terminal condition for an FBSDE on $[t'', t']$ with some $t'' \in [t, t']$ chosen again according to Remark 2.2.2.
We can now repeat this argument going to the left and conclude the proof using small interval induction (backward version). □

Remark 2.5.6. The requirement $(X, Y, Z) \in \mathbb{G}_0$, which means

$$\sup_{s\in[t,T]} \mathbb{E}_{0,\infty}[|X_s|^2] + \sup_{s\in[t,T]} \mathbb{E}_{0,\infty}[|Y_s|^2] + \mathbb{E}_{0,\infty}\left[\int_t^T |Z_s|^2 \,\mathrm{d}s\right] < \infty$$

is equivalent to

$$\sup_{s\in[t,T]} \mathbb{E}[|X_s|^2] + \sup_{s\in[t,T]} \mathbb{E}[|Y_s|^2] + \mathbb{E}\left[\int_t^T |Z_s|^2 \,\mathrm{d}s\right] < \infty$$

if $\mathcal{F}_0$ consists of null sets only.

Now, we want to explore how large the interval $[t, T]$ can be chosen, such that we still have (weakly regular) decoupling fields on this interval. It is natural to work with the following definition.

Definition 2.5.7. We define the *maximal interval* $I_{\max} \subseteq [0, T]$ for $(\xi, (\mu, \sigma, f))$ as the union of all intervals $[t, T] \subseteq [0, T]$, such that there exists a weakly regular decoupling field u on $[t, T]$.

Clearly, $I_{\max}$ is an interval. If it is not empty it contains T, which in this case is also its right boundary. Unfortunately the maximal interval might very well be open to the left. Therefore, we need to make our notions more precise in the following definitions:

Definition 2.5.8. Let $t < T$. We call a function $u : (t, T]\times\mathbb{R}^n \to \mathbb{R}^m$ a decoupling field for $(\xi, (\mu, \sigma, f))$ on $(t, T]$ if u restricted to $[t', T]$ is a decoupling field for all $t' \in (t, T]$.

Definition 2.5.9. Let $t < T$. We call a decoupling field u on $(t, T]$ *weakly regular* if u restricted to $[t', T]$ is weakly regular for all $t' \in (t, T]$.

Definition 2.5.10. Let $t < T$. We call a decoupling field u on $(t, T]$ *strongly regular* if u restricted to $[t', T]$ is strongly regular for all $t' \in (t, T]$.

Now, we can show the main result of this chapter.

Theorem 2.5.11 (Global existence in weak form)**.** *Let μ, σ, f, ξ be as in Theorem 2.2.1. Then there exists a unique weakly regular decoupling field u on $I_{\max}$. This u is even strongly regular.*
Furthermore, either $I_{\max} = [0, T]$ or $I_{\max} = (t_{\min}, T]$ where $0 \le t_{\min} < T$.

Proof. Let $t \in I_{\max}$. According to the definition of $I_{\max}$ there exists a decoupling field $\check{u}^{(t)}$ on $[t, T]$ satisfying $L_{\check{u}^{(t)},x} < L^{-1}_{\sigma,z}$ and $\sup_{s\in[t,T]} \|\check{u}^{(t)}(s, \cdot, 0)\|_\infty < \infty$. There is only one such $\check{u}^{(t)}$ by Corollary 2.5.3. Furthermore, for $t, t' \in I_{\max}$ the functions $\check{u}^{(t)}$ and $\check{u}^{(t')}$ coincide on $[t \vee t', T]$ again according to Corollary 2.5.3.
Define $u(t, \cdot) := \check{u}^{(t)}(t, \cdot)$ for all $t \in I_{\max}$. This function u is a decoupling field on $[t, T]$ since it coincides with $\check{u}^{(t)}$ on $[t, T]$. Therefore, u is a decoupling field on the whole interval $I_{\max}$ and satisfies $L_{u|_{[t,T]},x} < L^{-1}_{\sigma,z}$, $\sup_{s\in[t,T]} \|u(s, \cdot, 0)\|_\infty < \infty$ for all $t \in I_{\max}$.
Uniqueness of u follows directly from Corollary 2.5.3 applied to every interval $[t, T] \subseteq I_{\max}$.
Furthermore, u is strongly regular on $[t, T]$ for all $t \in I_{\max}$ because of Corollary 2.5.4.
Regarding the structure of $I_{\max}$: To prove the claim note that $I_{\max} = [t, T]$ with $t \in (0, T]$ is not possible: Assume otherwise. According to the first part of the current theorem there must be a decoupling field u on $[t, T]$ s.t. $L_{u,x} < L^{-1}_{\sigma,z}$ and $\sup_{s\in[t,T]} \|u(s, \cdot, 0)\|_\infty < \infty$. However, then we could use $u(t, \cdot)$ as a terminal condition to extend u a little bit to the left using Theorem 2.2.1 and Lemma 2.1.2, thereby contradicting the definition of $I_{\max}$. ☐

By *global existence in strong form* we mean the above weak global existence together with $I_{\max} = [0, T]$. Unfortunately the "bad" case $I_{\max} = (t_{\min}, T]$ is possible and is even more common. The following result basically says that this case can only occur if there is an "explosion" in the spatial derivative of u as we approach the lower boundary $t_{\min}$. By "explosion" we mean reaching the "forbidden" value $L^{-1}_{\sigma,z}$ which is ∞ in many applications.

Lemma 2.5.12. *Let μ, σ, f, ξ be as in Theorem 2.2.1. If $I_{\max} = (t_{\min}, T]$, then*

$$\lim_{t \downarrow t_{\min}} L_{u(t,\cdot),x} = L_{\sigma,z}^{-1},$$

where u is the unique decoupling field on $I_{\max}$.

Proof. This can be shown by contradiction. Clearly, $L_{u(t,\cdot),x} < L_{\sigma,z}^{-1}$ for all $t \in I_{\max}$. Assume in fact that we can select times $t_n \downarrow t_{\min}$ as $n \to \infty$ such that

$$\sup_{n \in \mathbb{N}} L_{u(t_n,\cdot),x} < L_{\sigma,z}^{-1}.$$

Considering $u(t_n, \cdot)$ as a possible terminal condition for the time t_n, we can choose $h > 0$ according to Remark 2.2.2. This h can be chosen independently of n precisely because of the above assumption. Now, choose n large enough to have $t_n - t_{\min} < h$. Hence, u can be extended to the left using Theorem 2.2.1 and Lemma 2.1.2 to a weakly regular decoupling field on an interval $[(t_n - h) \vee 0, T]$, beyond $t_{\min}$, thereby contradicting the definition of $I_{\max}$. □

Lemma 2.5.12 serves as a blueprint to show strong global existence in those cases in which it is suspected to hold. Let us describe the different steps.

1. Assume indirectly that $I_{\max} = [0, T]$ does not hold, which implies $I_{\max} = (t_{\min}, T]$. Choose arbitrary $t \in (t_{\min}, T]$, $x \in \mathbb{R}^n$ and consider the corresponding FBSDE.

2. Differentiate the FBSDE w.r.t. x. This is possible because of strong regularity of u (Theorem 2.5.11). We obtain joint dynamics of $\frac{\mathrm{d}}{\mathrm{d}x}X$, $\frac{\mathrm{d}}{\mathrm{d}x}Y$, $\frac{\mathrm{d}}{\mathrm{d}x}Z$.

3. Using Itô's formula deduce the dynamics of $\frac{\mathrm{d}}{\mathrm{d}x}Y_s(\frac{\mathrm{d}}{\mathrm{d}x}X_s)^{-1}$. This process can be expected to coincide with $u_x(s, X_s)$, as a consequence of the chain rule applied to the decoupling condition $Y_s = u(s, X_s)$.

4. Using the dynamics of $u_x(s, X_s)$ show that its modulus can be bounded away from $L_{\sigma,z}^{-1}$ independently of t, x, s, ω. This contradicts Lemma 2.5.12 and, therefore, $I_{\max} = [0, T]$ must hold.

This blueprint can be referred to as the *method of decoupling fields* to show global existence of solutions to FBSDEs (note also Corollary 2.5.5 at this point).

2.5.1 The Markovian case

An FBSDE given by μ, σ, f, ξ is said to be *Markovian*, if these four functions are deterministic, i.e. depend on t, x, y, z only (and not on ω).
In the Markovian case we can somewhat relax the Lipschitz continuity assumption and still obtain local existence together with uniqueness. It will allow us to treat problems with generators which are arbitrary polynomials in Y, Z. This will considerably expand the scope of potential applications.
What makes the Markovian case so special is the property

$$"Z_s = u_x(s, X_s) \cdot \sigma(s, X_s, Y_s, Z_s)"$$

which comes from the fact that u will also be deterministic. This property allows us to bound Z by a constant for instance if we assume that σ is uniformly bounded. This is a common trick (e.g. [Ric11], [Ric12]). We will be able to relax this boundedness of σ however.
The above relationship can be seen as a consequence of the Itô formula, applied to $u(s, X_s)$, assuming that u is smooth enough. However, under our assumptions u will not have sufficient smoothness.
In the literature sometimes Malliavin's calculus is used to deduce such a relationship (e.g. [dR10]).

However, we will follow simpler arguments. Note that under our assumptions u is only weakly differentiable in x, such that $u_x(s,\cdot)$ is only unique up to null sets. At the same time the distribution of X_s does not have to be absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}^n$. Therefore, $u_x(s,X_s)$ might not be properly defined and so we will not actually try to show $Z_s = u_x(s,X_s)\cdot\sigma(s,X_s,Y_s,Z_s)$. We are only interested in bounding Z.
Finally let us remark that in the Markovian case the FBSDEs we consider are closely connected with a major class of quasi-linear parabolic partial differential equations via the Feynman-Kac formula. A decoupling field u can be seen as a type of solution to such a PDE. Notice that we are able to develop an existence and uniqueness theory under mild analytic assumptions. Basically we only need Lipschitz or even just local Lipschitz continuity (as we will see later) of the parameters involved. Under our assumptions u will only be Lipschitz continuous in space and continuous as a function of time and space. But in order to write down the classical PDE u has to be differentiable in time (at least weakly) and twice differentiable in space, which requires more restrictive assumptions. Thus, we have a weaker form of solvability allowing us to have existence and uniqueness for a more general class of problems. Also, note that with this stochastic interpretation of second order PDEs we are able to implement a rather explicit construction. So far we have either used Picard iterations and Banach's fixed point theorem or a technique of "gluing together" decoupling fields on adjacent intervals using Lemma 2.1.2. This gives us a better control of the objects constructed and can potentially serve as basis for numerical methods.

First, let us prove the following statement.

Lemma 2.5.13. *Let μ,σ,f,ξ be as in Theorem 2.2.1 and assume in addition that they are deterministic. Assume further that we have a weakly regular decoupling field u on an interval $[t,T]$. Then u is deterministic in the sense that it has a modification which is a function of $(r,x)\in[t,T]\times\mathbb{R}^n$ only.*

Proof. u is strongly regular according to Corollary 2.5.4. Let $s\in[t,T]$ and $x\in\mathbb{R}^n$. Let X,Y,Z be the associated processes on $[s,T]$ satisfying the forward and the backward equation together with the decoupling condition. In particular, X,Y,Z satisfy the FBSDE with terminal condition $Y_T=\xi(X_T)$. We can also assume that X,Y are continuous in time. Furthermore, they satisfy (2.27).
We can decompose $\Omega=\Omega_{[0,s]}\times\Omega_{(s,T]}$, where the first component contains only the information about $\mathcal{F}_0$ and the noise on $[0,s]$, while the second component only contains the information about the noise on $[s,T]$, i.e. is generated by $(W_{s+h}-W_s)_{h\in[0,T-s]}$ and is independent of $\mathcal{F}_0$ and the noise until t. We now fix $\underline{\omega}_1\in\Omega_{[0,s]}$ and make the following observation which holds for almost every $\underline{\omega}_1\in\Omega_{[0,s]}$:
Since μ,σ,f are deterministic and $(W_{s+h}-W_s)_{h\in[0,T-s]}$ does not depend on ω_1, the processes $\tilde{X},\tilde{Y},\tilde{Z}$ obtained from X,Y,Z by replacing $\omega=(\omega_1,\omega_2)$ with $(\underline{\omega}_1,\omega_2)$ will still satisfy the FBSDE for a.a. $\omega_2\in\Omega$. Therefore, due to Corollary 2.5.5 these new processes $\tilde{X},\tilde{Y},\tilde{Z}$, which depend on ω through ω_2 only, must coincide with X,Y,Z.
In particular, $u(s,\cdot,x)=u(s,(\underline{\omega}_1,\cdot),x)$ a.s. for a.a. $\underline{\omega}_1\in\Omega_{[0,s]}$. Since for every $x\in\mathbb{R}^n$ the random variable $u(s,(\underline{\omega}_1,\cdot),x)$ is $\mathcal{F}_s$-measurable, it is independent of $(W_{s+h}-W_s)_{h\in[0,T-s]}$ and since it obviously does not depend on ω_1 either it must be a.s. constant. Therefore, $u(s,\cdot,x)=\mathbb{E}[u(s,\cdot,x)]$ a.s. as well.
Now, let $\tilde{\Omega}\in\mathcal{F}$ be such that

- $u(s,\omega,x)=\mathbb{E}[u(s,\cdot,x)]$ for all $x\in\mathbb{Q}^n$ and $\omega\in\tilde{\Omega}$,
- $u(s,\omega,\cdot)$ is Lipschitz-continuous for all $\omega\in\tilde{\Omega}$,
- $\mathbb{P}(\tilde{\Omega})=1$.

The second requirement can be fulfilled due to weak regularity of u, the first can be fulfilled because $\mathbb{Q}^n$ is a countable set. The above three properties imply that $u(s,\omega,\cdot)$ is equal to the deterministic Lipschitz-continuous function $x\longmapsto\mathbb{E}[u(s,\cdot,x)]$ for almost all $\omega\in\Omega$. □

As an application we show the following fundamental result.

Lemma 2.5.14. *Let μ, σ, f, ξ be as in Theorem 2.2.1 and assume in addition that they are deterministic. Let u be a weakly regular decoupling field on an interval $[t, T]$. Choose $t_1 < t_2$ from $[t, T]$ and an initial condition X_{t_1}. Then the corresponding Z satisfies $\|Z\|_\infty \leq L_{u,x} \cdot \|\sigma\|_\infty$.*
If $\|Z\|_\infty < \infty$, we also have

$$\|Z\|_\infty \leq \frac{L_{u,x} \cdot \|\sigma(\cdot,\cdot,\cdot,0)\|_\infty}{1 - L_{u,x}L_{\sigma,z}} \quad \text{and} \quad \|Z\|_\infty \leq \frac{L_{u,x}\left(\|\sigma(\cdot,\cdot,0,0)\|_\infty + L_{\sigma_0,y}\|Y\|_\infty\right)}{1 - L_{u,x}L_{\sigma,z}},$$

where $L_{\sigma_0,y}$ is the Lipschitz constant of $\sigma(\cdot,\cdot,\cdot,0)$ w.r.t. $y \in \mathbb{R}^m$.

Proof. u is deterministic according to Lemma 2.5.13 and strongly regular according to Corollary 2.5.4. We assume from now on that u is a function of $(r, x) \in [t, T] \times \mathbb{R}^n$ only.

CASE 1: Assume $X_{t_1} = x \in \mathbb{R}^n$:

Due to strong regularity the associated X, Y, Z on $[t_1, t_2]$ solving the forward equation, the backward equation and the decoupling condition are unique. We assume without loss of generality that $t_2 = T$.
Notice $\lim_{s\downarrow s'} \frac{1}{s-s'}\int_{s'}^s Z_r(\omega)\,\mathrm{d}r = Z_{s'}(\omega)$ for almost all $(\omega, s') \in \Omega \times [t, T]$ due to the fundamental Theorem of Lebesgue integral calculus. The same holds for the expressions $\frac{1}{s-s'}\int_{s'}^s f(r, X_r, Y_r, Z_r)W_r^\top\,\mathrm{d}r$ and $\frac{1}{s-s'}\int_{s'}^s f(r, X_r, Y_r, Z_r)\,\mathrm{d}r$.
Also, $\mathbb{E}[|Z_{s'}|^2] < \infty$ for almost all $s' \in [t, T]$ due to $\mathbb{E}\left[\int_t^T |Z_r|^2\,\mathrm{d}r\right] < \infty$. The same holds for the expression $\mathbb{E}[|f(s', X_{s'}, Y_{s'}, Z_{s'})|^2]$.
Choose an $s' \in [t_1, T)$ with

- $\lim_{s\downarrow s'} \frac{1}{s-s'}\int_{s'}^s Z_r\,\mathrm{d}r = Z_{s'}$ a.s.,
- $\lim_{s\downarrow s'} \frac{1}{s-s'}\int_{s'}^s f(r, X_r, Y_r, Z_r)W_r^\top\,\mathrm{d}r = f(s', X_{s'}, Y_{s'}, Z_{s'})W_{s'}^\top$ a.s.,
- $\lim_{s\downarrow s'} \frac{1}{s-s'}\int_{s'}^s f(r, X_r, Y_r, Z_r)\,\mathrm{d}r = f(s', X_{s'}, Y_{s'}, Z_{s'})$ a.s.,
- $\mathbb{E}[|Z_{s'}|^2] + \mathbb{E}[|f(s', X_{s'}, Y_{s'}, Z_{s'})|^2] < \infty$.

Almost all $s' \in [t_1, T)$ will have these properties.
For every $(\mathcal{F}_s)_{s\in[s',T]}$ - stopping time $\tau : \Omega \to (s', T]$ we have according to the product rule applied to the backward equation:

$$\begin{aligned} &Y_\tau(W_\tau - W_{s'})^\top = \\ &= \int_{s'}^\tau Y_r\,\mathrm{d}W_r^\top + \int_{s'}^\tau f(r, X_r, Y_r, Z_r)(W_r - W_{s'})^\top\,\mathrm{d}r + \int_{s'}^\tau Z_r\,\mathrm{d}W_r(W_r - W_{s'})^\top + \int_{s'}^\tau Z_r\,\mathrm{d}r. \end{aligned} \tag{2.29}$$

τ can be chosen in such a way that

- $\left(\int_{s'}^{\tau\wedge s} Y_r\,\mathrm{d}W_r^\top\right)_{s\in[s',T]}$ and $\left(\int_{s'}^{\tau\wedge s} Z_r\,\mathrm{d}W_r(W_r - W_{s'})^\top\right)_{s\in[s',T]}$ are uniformly integrable martingales,
- $\left|\frac{1}{s-s'}\int_{s'}^{\tau\wedge s} Z_r\,\mathrm{d}r\right| \leq |Z_{s'}| + 1$ a.s.,
- $\left|\frac{1}{s-s'}\int_{s'}^{\tau\wedge s} f(r, X_r, Y_r, Z_r)\,\mathrm{d}r\right| \leq |f(s', X_{s'}, Y_{s'}, Z_{s'})| + 1$ a.s.,
- $\left|\frac{1}{s-s'}\int_{s'}^{\tau\wedge s} f(r, X_r, Y_r, Z_r)W_r^\top\,\mathrm{d}r\right| \leq |f(s', X_{s'}, Y_{s'}, Z_{s'})W_{s'}^\top| + 1$ a.s., for all $s \in (s', T]$.

The first statement should be clear. From the remaining three we only discuss the first in more detail: Define $U_s := \frac{1}{s-s'}\int_{s'}^{s} Z_r \,\mathrm{d}r$ for $s \in (s', T]$ and set $U_{s'} := Z_{s'}$. U is an a.s.-continuous and adapted process starting at $Z_{s'}$. If we choose τ such that the stopping occurs when $|U|$ reaches $|Z_{s'}|+1$, then $\tau > s'$ will hold and also $\left|\frac{1}{s-s'}\int_{s'}^{\tau\wedge s} Z_r \,\mathrm{d}r\right| = \left|\frac{\tau\wedge s - s'}{s-s'} U_{\tau\wedge s}\right| \leq |U_{\tau\wedge s}| \leq |Z_{s'}|+1$ a.s., for $s \in (s', T]$. If we stop earlier, i.e. choose a smaller stopping time, the bound still holds as long as $\tau > s'$.
Now, we apply conditional expectations on both sides of (2.29), divide by $s-s'$ and pass to the limit applying dominated convergence along the way:

$$\begin{aligned}
&\lim_{s\downarrow s'} \mathbb{E}\left[\frac{1}{s-s'} Y_{\tau\wedge s}(W_{\tau\wedge s} - W_{s'})^\top \middle| \mathcal{F}_{s'}\right] = \\
&\quad = \lim_{s\downarrow s'} \mathbb{E}\left[\frac{1}{s-s'}\int_{s'}^{\tau\wedge s} f(r, X_r, Y_r, Z_r)(W_r - W_{s'})^\top \,\mathrm{d}r \middle| \mathcal{F}_{s'}\right] + \lim_{s\downarrow s'} \mathbb{E}\left[\frac{1}{s-s'}\int_{s'}^{\tau\wedge s} Z_r \,\mathrm{d}r \middle| \mathcal{F}_{s'}\right] = \\
&\qquad = \mathbb{E}\left[\lim_{s\downarrow s'} \frac{1}{s-s'}\int_{s'}^{\tau\wedge s} f(r, X_r, Y_r, Z_r) W_r^\top \,\mathrm{d}r \middle| \mathcal{F}_{s'}\right] - \\
&\quad - \mathbb{E}\left[\lim_{s\downarrow s'} \frac{1}{s-s'}\int_{s'}^{\tau\wedge s} f(r, X_r, Y_r, Z_r) W_{s'}^\top \,\mathrm{d}r \middle| \mathcal{F}_{s'}\right] + \mathbb{E}\left[\lim_{s\downarrow s'} \frac{1}{s-s'}\int_{s'}^{\tau\wedge s} Z_r \,\mathrm{d}r \middle| \mathcal{F}_{s'}\right] = \\
&\qquad = f(s', X_{s'}, Y_{s'}, Z_{s'}) W_{s'}^\top - f(s', X_{s'}, Y_{s'}, Z_{s'}) W_{s'}^\top + Z_{s'} = Z_{s'},
\end{aligned}$$

where we used $\tau(\omega) \wedge s = s$ for $s \in (s', T]$ small enough.
This of course implies

$$\lim_{s\downarrow s'} \left|\mathbb{E}\left[\frac{1}{s-s'} Y_{\tau\wedge s}(W_{\tau\wedge s} - W_{s'})^\top \middle| \mathcal{F}_{s'}\right]\right| = |Z_{s'}|.$$

At the same time using triangle inequality, the decoupling condition and $\tau \wedge s > s'$ for all $s \in (s', T]$:

$$\begin{aligned}
&\left|\mathbb{E}\left[\frac{1}{s-s'} Y_{\tau\wedge s}(W_{\tau\wedge s} - W_{s'})^\top \middle| \mathcal{F}_{s'}\right]\right| \leq \\
&\quad \leq \left|\mathbb{E}\left[\frac{1}{s-s'} Y_s (W_{\tau\wedge s} - W_{s'})^\top \middle| \mathcal{F}_{s'}\right]\right| + \left|\mathbb{E}\left[\frac{1}{s-s'} \left(Y_s - Y_{\tau\wedge s}\right)\left(W_{\tau\wedge s} - W_{s'}\right)^\top \middle| \mathcal{F}_{s'}\right]\right| = \\
&\quad = \left|\mathbb{E}\left[\frac{1}{s-s'} u(s, X_s)(W_{\tau\wedge s} - W_{s'})^\top \middle| \mathcal{F}_{s'}\right]\right| + \left|\mathbb{E}\left[\frac{1}{s-s'} \mathbb{E}\left[Y_s - Y_{\tau\wedge s} | \mathcal{F}_{\tau\wedge s}\right]\left(W_{\tau\wedge s} - W_{s'}\right)^\top \middle| \mathcal{F}_{s'}\right]\right|.
\end{aligned}$$

Let us now investigate the two summands separately:

First summand:

Using $\mathcal{F}_{s'}$ - measurability of $u(s, X_{s'})$ together with Cauchy-Schwarz inequality:

$$\begin{aligned}
&\left|\mathbb{E}\left[\frac{1}{s-s'} u(s, X_s)(W_{\tau\wedge s} - W_{s'})^\top \middle| \mathcal{F}_{s'}\right]\right| = \left|\mathbb{E}\left[\frac{1}{s-s'}\left(u(s, X_s) - u(s, X_{s'})\right)\left(W_{\tau\wedge s} - W_{s'}\right)^\top \middle| \mathcal{F}_{s'}\right]\right| \leq \\
&\leq \mathbb{E}\left[\frac{1}{s-s'}\left|\left(u(s, X_s) - u(s, X_{s'})\right)\left(W_{\tau\wedge s} - W_{s'}\right)^\top\right| \middle| \mathcal{F}_{s'}\right] \leq \\
&\qquad \leq \mathbb{E}\left[\frac{1}{s-s'}\left|u(s, X_s) - u(s, X_{s'})\right| \left|W_{\tau\wedge s} - W_{s'}\right| \middle| \mathcal{F}_{s'}\right] \leq \\
&\qquad\quad \leq \frac{1}{s-s'}\left(\mathbb{E}\left[\left|u(s, X_s) - u(s, X_{s'})\right|^2 \left|\mathcal{F}_{s'}\right.\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[|W_{\tau\wedge s} - W_{s'}|^2 | \mathcal{F}_{s'}\right]\right)^{\frac{1}{2}}.
\end{aligned}$$

We can further estimate this using $L_{u,x} < \infty$ and the forward equation to which the Minkowski inequality is applied:

$$\left|\mathbb{E}\left[\frac{1}{s-s'}u(s,X_s)(W_{\tau\wedge s}-W_{s'})^\top\middle|\mathcal{F}_{s'}\right]\right| \le \frac{1}{s-s'}L_{u,x}\left(\mathbb{E}\left[|X_s-X_{s'}|^2\,|\mathcal{F}_{s'}\right]\right)^{\frac{1}{2}}\sqrt{s-s'} \le$$

$$\le \frac{1}{\sqrt{s-s'}}L_{u,x}\left(\left(\mathbb{E}\left[\left|\int_{s'}^{s}\mu(r,X_r,Y_r,Z_r)\,\mathrm{d}r\right|^2\middle|\mathcal{F}_{s'}\right]\right)^{\frac{1}{2}}+\left(\mathbb{E}\left[\int_{s'}^{s}|\sigma(r,X_r,Y_r,Z_r)|^2\,\mathrm{d}r\middle|\mathcal{F}_{s'}\right]\right)^{\frac{1}{2}}\right)$$

$$\le L_{u,x}\left(\mathbb{E}\left[\int_{s'}^{s}|\mu(r,X_r,Y_r,Z_r)|^2\,\mathrm{d}r\middle|\mathcal{F}_{s'}\right]\right)^{\frac{1}{2}}+L_{u,x}\|\sigma(\cdot,X,Y,Z)\|_\infty,$$

where the first term converges a.s. to 0 for $s\to s'$. Note that all integrals are well-defined and finite due to strong regularity and the fact that μ,σ are as in Theorem 2.2.1. Also, note that we do not rule out $\|\sigma(\cdot,X,Y,Z)\|_\infty=\infty$ at this point.

Second summand:
Using the backward equation and Cauchy-Schwarz inequality:

$$\left|\mathbb{E}\left[\frac{1}{s-s'}\mathbb{E}\left[Y_s-Y_{\tau\wedge s}|\mathcal{F}_{\tau\wedge s}\right](W_{\tau\wedge s}-W_{s'})^\top\middle|\mathcal{F}_{s'}\right]\right| =$$

$$= \left|\mathbb{E}\left[\frac{1}{s-s'}\mathbb{E}\left[\int_{\tau\wedge s}^{s}f(r,X_r,Y_r,Z_r)\,\mathrm{d}r\middle|\mathcal{F}_{\tau\wedge s}\right](W_{\tau\wedge s}-W_{s'})^\top\middle|\mathcal{F}_{s'}\right]\right| \le$$

$$\le \frac{1}{s-s'}\left(\mathbb{E}\left[\left|\int_{\tau\wedge s}^{s}f(r,X_r,Y_r,Z_r)\,\mathrm{d}r\right|^2\middle|\mathcal{F}_{s'}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[|W_{\tau\wedge s}-W_{s'}|^2\,|\mathcal{F}_{s'}\right]\right)^{\frac{1}{2}}.$$

Using Cauchy-Schwarz inequality a second time together with $\tau\wedge s\le s$ we can estimate this value by:

$$\frac{1}{s-s'}\left(\mathbb{E}\left[(s-s')\int_{s'}^{s}|f(r,X_r,Y_r,Z_r)|^2\,\mathrm{d}r\middle|\mathcal{F}_{s'}\right]\right)^{\frac{1}{2}}\sqrt{s-s'} =$$

$$= \left(\mathbb{E}\left[\int_{s'}^{s}|f(r,X_r,Y_r,Z_r)|^2\,\mathrm{d}r\middle|\mathcal{F}_{s'}\right]\right)^{\frac{1}{2}},$$

which a.s. converges to 0 as $s\to s'$. Note that all integrals are well-defined and finite due to strong regularity and the fact that f is as in Theorem 2.2.1.

Conclusion:
Finally we have shown

$$|Z_{s'}| = \limsup_{s\downarrow s'}\left|\mathbb{E}\left[\frac{1}{s-s'}Y_{\tau\wedge s}(W_{\tau\wedge s}-W_{s'})^\top\middle|\mathcal{F}_{s'}\right]\right| \le L_{u,x}\|\sigma(\cdot,X,Y,Z)\|_\infty \le L_{u,x}\|\sigma\|_\infty \text{ a.s.}$$

Note that this argument works for almost all $s'\in[t,T]$, as mentioned in the beginning.

At the same instead of estimating $\|\sigma(\cdot,X,Y,Z)\|_\infty$ by simply $\|\sigma\|_\infty$, we could have estimated it by $\|\sigma(\cdot,X,Y,0)\|_\infty+L_{\sigma,z}\|Z\|_\infty$, which leads to

$$|Z_{s'}| \le L_{u,x}\cdot(\|\sigma(\cdot,X,Y,0)\|_\infty+L_{\sigma,z}\|Z\|_\infty) \qquad \text{a.s.}$$

for a.a. $s'\in[t,T]$, which means $\|Z\|_\infty\le L_{u,x}\|\sigma(\cdot,X,Y,0)\|_\infty+L_{u,x}L_{\sigma,z}\|Z\|_\infty$, or

$$\|Z\|_\infty \le \frac{L_{u,x}\cdot\|\sigma(\cdot,X,Y,0)\|_\infty}{1-L_{u,x}L_{\sigma,z}},$$

if Z is bounded up to a null set. Note that $L_{u,x}L_{\sigma,z} < 1$ due to weak regularity of u. Note also that $\|\sigma(\cdot, X, Y, 0)\|_\infty \le \|\sigma(\cdot,\cdot,\cdot,0)\|_\infty$ and $\|\sigma(\cdot, X, Y, 0)\|_\infty \le \|\sigma(\cdot,\cdot,0,0)\|_\infty + L_{\sigma 0,y}\|Y\|_\infty$, due to Lipschitz continuity of σ in the last three components. ✓

CASE 2: For $t_1 \in [t,T]$ and $t_2 \in [t_1,T]$ let X_{t_1} be any $\mathcal{F}_{t_1}$ - measurable random variable serving as the initial condition. Consider the corresponding X, Y, Z on $[t_1, t_2]$.
We can decompose $\Omega = \Omega_{[0,t_1]} \times \Omega_{(t_1,T]}$, where the first component contains only the information about $\mathcal{F}_0$ and the noise on $[0,t_1]$, while the second component is generated by $(W_{t_1+h} - W_{t_1})_{h\in[0,T-t_1]}$. If we fix the first component $\underline{\omega}_1 \in \Omega_{[0,t_1]}$, the initial value $X_{t_1}(\underline{\omega}_1,\cdot)$ is an a.s.-constant while the processes $X.(\underline{\omega}_1,\cdot), Y.(\underline{\omega}_1,\cdot), Z.(\underline{\omega}_1,\cdot)$ still solve the forward equation, the backward equation and the decoupling condition for almost every $\underline{\omega}_1 \in \Omega_{[0,t_1]}$, for every $s \in [t_1,t_2]$. This is because μ, σ, f, u do not depend on ω. Note that it is sufficient to consider only countably many s due to continuity of X, Y and right-continuity of u according to Lemma 2.1.3.

This already implies $\|Z(\underline{\omega}_1,\cdot)\|_\infty \le L_{u,x}\cdot\|\sigma\|_\infty$ using CASE 1.
Since this argument works for almost all $\underline{\omega}_1$, we have $\|Z\|_\infty \le L_{u,x}\cdot\|\sigma\|_\infty$.
Similarly we get $\|Z(\underline{\omega}_1,\cdot)\|_\infty \le L_{u,x}\|\sigma(\cdot,X,Y,0)\|_\infty + L_{u,x}L_{\sigma,z}\|Z(\underline{\omega}_1,\cdot)\|_\infty$ for almost all $\underline{\omega}_1$, which implies $\|Z\|_\infty \le L_{u,x}\|\sigma(\cdot,X,Y,0)\|_\infty + L_{u,x}L_{\sigma,z}\|Z\|_\infty$. □

Next we investigate the continuity of u as a function of time and space.

Lemma 2.5.15. *Assume that μ, σ, f have linear growth in (x, y) in the sense*

$$(|\mu| + |\sigma| + |f|)(t,\omega,x,y,z) \le C(1 + |x| + |y|) \quad \forall (t,x,y,z) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d},$$

for a.a. $\omega \in \Omega$, where $C \in [0,\infty)$ is some constant.
If u is a strongly regular and deterministic decoupling field to $(\xi,(\mu,\sigma,f))$ on an interval $[t,T]$, then u is continuous in the sense that it has a modification which is a continuous function on $[t,T]\times\mathbb{R}^n$.

Proof. Choose any $t_1 < t_2$ from the interval $[t,T]$ and some $x \in \mathbb{R}^n$ as initial value. Consider the corresponding unique X, Y, Z fulfilling the forward and backward equations together with the decoupling condition. We assume without loss of generality that X and Y are continuous in time. Because of the forward equation together with the linear growth condition for μ, σ and strong regularity we can show $\mathbb{E}[\sup_{s\in[t_1,t_2]}|X_s|^2] < \infty$: We have

$$\mathbb{E}\left[\left(\int_{t_1}^{t_2}|\mu(r,X_r,Y_r,Z_r)|\,\mathrm{d}r\right)^2\right] \le \mathbb{E}\left[(t_2-t_1)\int_{t_1}^{t_2}|\mu(r,X_r,Y_r,Z_r)|^2\,\mathrm{d}r\right] < \infty,$$

where we used Cauchy-Schwarz inequality plus (2.27) at the end. Similarly

$$\mathbb{E}\left[\sup_{s\in[t_1,t_2]}\left|\int_{t_1}^{s}\sigma(r,X_r,Y_r,Z_r)\,\mathrm{d}W_r\right|^2\right] \le 4\mathbb{E}\left[\left|\int_{t_1}^{t_2}\sigma(r,X_r,Y_r,Z_r)\,\mathrm{d}W_r\right|^2\right] =$$

$$= 4\mathbb{E}\left[\int_{t_1}^{t_2}|\sigma(r,X_r,Y_r,Z_r)|^2\,\mathrm{d}r\right] < \infty,$$

where we used Doob's inequality and Itô isometry. ✓

Using Itô formula and the forward equation we have

$$|X_s|^2 = x^2 + 2\int_{t_1}^{s} X_r^\top\mu(r,X_r,Y_r,Z_r)\,\mathrm{d}r + 2\int_{t_1}^{s} X_r^\top\sigma(r,X_r,Y_r,Z_r)\,\mathrm{d}W_r + \int_{t_1}^{s}|\sigma|^2(r,X_r,Y_r,Z_r)\,\mathrm{d}r$$

a.s. for all $s \in [t_1,t_2]$. The stochastic integral in the middle might be just a local martingale, so we have to work with a localizing sequence $\tau_n \uparrow t_2$ of $[t_1,t_2]$ - valued stopping times such that

$\int_{t_1}^{\cdot\wedge\tau_n} X_r^\top \sigma(r, X_r, Y_r, Z_r)\,\mathrm{d}W_r$ is a uniformly integrable martingale. We have for every $n \in \mathbb{N}$:

$$|X_{s\wedge\tau_n}|^2 = x^2 + 2\int_{t_1}^{s} \mathbf{1}_{\{r\le\tau_n\}} X_r^\top \mu(r, X_r, Y_r, Z_r)\,\mathrm{d}r + 2\int_{t_1}^{s} \mathbf{1}_{\{r\le\tau_n\}} X_r^\top \sigma(r, X_r, Y_r, Z_r)\,\mathrm{d}W_r + \\ + \int_{t_1}^{s} \mathbf{1}_{\{r\le\tau_n\}} |\sigma|^2(r, X_r, Y_r, Z_r)\,\mathrm{d}r$$

a.s. for all $s \in [t_1, t_2]$. Applying the expectation on both sides of the above equation and using the required linear growth condition for μ, σ, f we obtain for $n \in \mathbb{N}$:

$$\mathbb{E}\left[|X_{s\wedge\tau_n}|^2\right] \le x^2 + \int_{t_1}^{s} C_1\left(1 + \mathbb{E}[\mathbf{1}_{\{r\le\tau_n\}}|X_r|^2] + \mathbb{E}[\mathbf{1}_{\{r\le\tau_n\}}|Y_r|^2]\right)\,\mathrm{d}r,$$

with some real constant $C_1 > 0$ depending on C only. Letting $n \to \infty$ we have using dominated convergence with dominating function $\sup_{s\in[t_1,t_2]} |X_s|^2$:

$$\mathbb{E}\left[|X_s|^2\right] \le x^2 + \int_{t_1}^{s} C_1\left(1 + \mathbb{E}[|X_r|^2] + \mathbb{E}[|Y_r|^2]\right)\,\mathrm{d}r, \tag{2.30}$$

The decoupling condition implies

$$|Y_r| = |u(r, X_r)| \le \sup_{s\in[t,T]} \|u(s,\cdot,0)\|_\infty + L_{u,x}|X_r| \quad \text{a.s.}, \tag{2.31}$$

for all $r \in [t_1, t_2]$, which together with (2.30) implies

$$\mathbb{E}\left[|X_s|^2\right] \le x^2 + \int_{t_1}^{s} C_2\left(1 + \mathbb{E}[|X_r|^2]\right)\,\mathrm{d}r \qquad \forall s \in [t_1, t_2],$$

with some real constant $C_2 > 0$ which depends on C_1, $\sup_{s\in[t,T]} \|u(s,\cdot,0)\|_\infty$ and $L_{u,x}$. Using Gronwall's lemma we now obtain

$$\sup_{s\in[t_1,t_2]} \mathbb{E}\left[|X_s|^2\right] \le \left(x^2 + (t_2 - t_1)C_2\right) e^{C_2(t_2-t_1)} < \infty. \tag{2.32}$$

Also, using (2.31) we have

$$\sup_{s\in[t_1,t_2]} \mathbb{E}\left[|Y_s|^2\right] \le C_3\left(1 + \sup_{s\in[t_1,t_2]} \mathbb{E}\left[|X_s|^2\right]\right), \tag{2.33}$$

with some real constant $C_3 > 0$ which depends on $\sup_{s\in[t,T]} \|u(s,\cdot,0)\|_\infty$ and $L_{u,x}$.

We use from now on that the decoupling field u is a function on $[t,T] \times \mathbb{R}^n$.
We claim that u satisfies

$$|u(t_1, x) - u(t_2, x)| \le \tilde{C}(1 + |x|)|t_1 - t_2|^{\frac{1}{2}} \qquad \forall t_1, t_2 \in [t,T], x \in \mathbb{R}^n \tag{2.34}$$

with some real constant $\tilde{C} > 0$:

Assume without loss of generality that $t_1 < t_2$. We use the triangle inequality together with the decoupling condition $Y_s = u(s, X_s)$:

$$|u(t_2, x) - u(t_1, x)| \le |u(t_2, x) - \mathbb{E}[u(t_2, X_{t_2})]| + |\mathbb{E}[u(t_2, X_{t_2})] - u(t_1, x)| = \\ = |\mathbb{E}[u(t_2, x) - u(t_2, X_{t_2})]| + |\mathbb{E}[Y_{t_2} - Y_{t_1}]| \le L_{u,x}\mathbb{E}[|X_{t_2} - x|] + |\mathbb{E}[Y_{t_2} - Y_{t_1}]|.$$

Let us investigate the terms $\mathbb{E}[|X_{t_2} - x|]$ and $|\mathbb{E}[Y_{t_2} - Y_{t_1}]|$ separately:

FIRST TERM:

Use the forward equation, the Minkowski inequality, the Cauchy-Schwarz inequality and the Itô isometry to obtain:

$$\sqrt{\mathbb{E}[|X_{t_2} - x|^2]} \leq \left(\mathbb{E}\left[\left|\int_{t_1}^{t_2} \mu(r, X_r, Y_r, Z_r)\,\mathrm{d}r\right|^2\right]\right)^{\frac{1}{2}} + \left(\mathbb{E}\left[\left|\int_{t_1}^{t_2} \sigma(r, X_r, Y_r, Z_r)\,\mathrm{d}W_r\right|^2\right]\right)^{\frac{1}{2}} \leq$$
$$\leq \sqrt{t_2 - t_1}\left(\mathbb{E}\left[\int_{t_1}^{t_2} |\mu(r, X_r, Y_r, Z_r)|^2\,\mathrm{d}r\right]\right)^{\frac{1}{2}} + \left(\mathbb{E}\left[\int_{t_1}^{t_2} |\sigma|^2(r, X_r, Y_r, Z_r)\,\mathrm{d}r\right]\right)^{\frac{1}{2}}.$$

Again all integrals are well-defined and finite due to strong regularity. Using the linear growth condition required for μ, σ together with (2.33) we can further estimate this from above by the value

$$\sqrt{t_2 - t_1}\left(\mathbb{E}\left[\int_{t_1}^{t_2} C_4\left(1 + |X_r|^2 + |Y_r|^2\right)\,\mathrm{d}r\right]\right)^{\frac{1}{2}} + \left(\mathbb{E}\left[\int_{t_1}^{t_2} C_4\left(1 + |X_r|^2 + |Y_r|^2\right)\,\mathrm{d}r\right]\right)^{\frac{1}{2}} \leq$$
$$\leq C_5\left(1 + \sqrt{\sup_{r\in[t_1,t_2]} \mathbb{E}[|X_r|^2]}\right)\sqrt{t_2 - t_1}\sqrt{t_2 - t_1} + C_5\left(1 + \sqrt{\sup_{r\in[t_1,t_2]} \mathbb{E}[|X_r|^2]}\right)\sqrt{t_2 - t_1}$$

where $C_4 \in (0, \infty)$ is some constant, which depends on C, while $C_5 \in (0, \infty)$ is a constant depending on C_4 and C_3.

SECOND TERM:

Use the backward equation together with strong regularity to obtain:

$$|\mathbb{E}[Y_{t_2} - Y_{t_1}]| \leq \left|\int_{t_1}^{t_2} \mathbb{E}[f(r, X_r, Y_r, Z_r)]\,\mathrm{d}r\right| \leq \int_{t_1}^{t_2} \mathbb{E}[|f(r, X_r, Y_r, Z_r)|]\,\mathrm{d}r,$$

which using the linear growth condition required for f together with (2.33) can be controlled by

$$\int_{t_1}^{t_2} C\left(1 + \mathbb{E}[|X_r|] + \mathbb{E}[|Y_r|]\right)\,\mathrm{d}r \leq C_6\left(1 + \sqrt{\sup_{r\in[t_1,t_2]} \mathbb{E}[|X_r|^2]}\right)(t_2 - t_1),$$

where C_6 is some real constant depending only on C and C_3: In the last step we used Cauchy-Schwarz inequality together with (2.33).

CONCLUSION:

We have finally shown

$$|u(t_2, x) - u(t_1, x)| \leq C_7\left(1 + \sqrt{\sup_{r\in[t_1,t_2]} \mathbb{E}[|X_r|^2]}\right)\sqrt{t_2 - t_1},$$

with some real constant C_7 depending only on C_5, C_6 and T. Now, the claim (2.34) follows directly from (2.32). ✓

$L_{u,x} < \infty$ implies that u is Lipschitz-continuous in $x \in \mathbb{R}^n$. Together with the Hölder continuity property of (2.34) for the dependence on time, continuity of u as a function of $(s, x) \in [t, T] \times \mathbb{R}^n$ follows easily from

$$|u(t_2, x_2) - u(t_1, x_1)| \leq |u(t_2, x_2) - u(t_1, x_2)| + |u(t_1, x_2) - u(t_1, x_1)| \leq$$
$$\leq \tilde{C}(1 + |x_2|)|t_2 - t_1|^{\frac{1}{2}} + L_{u,x}|x_2 - x_1| \quad \forall t_1, t_2 \in [t, T],\ x_1, x_2 \in \mathbb{R}^n$$

and the proof is complete. □

Lemma 2.5.14 indicates potential boundedness of Z in the Markovian case. This motivates the following definition, which will allow us to develop a theory for non-Lipschitz problems:

Definition 2.5.16. Let $\xi : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ be measurable and let $t \in [0, T]$.
We call a function $u : [t, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ with $u(T, \omega, \cdot) = \xi(\omega, \cdot)$ for a.a. $\omega \in \Omega$ a *Markovian decoupling field* for $(\xi, (\mu, \sigma, f))$ on $[t, T]$ if for all $t_1, t_2 \in [t, T]$ with $t_1 \le t_2$ and any $\mathcal{F}_{t_1}$ - measurable $X_{t_1} : \Omega \to \mathbb{R}^n$ there exist progressive processes X, Y, Z on $[t_1, t_2]$ such that

- $X_s = X_{t_1} + \int_{t_1}^s \mu(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_{t_1}^s \sigma(r, X_r, Y_r, Z_r)\,\mathrm{d}W_r$ a.s.,
- $Y_s = Y_{t_2} - \int_s^{t_2} f(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^{t_2} Z_r\,\mathrm{d}W_r$ a.s.,
- $Y_s = u(s, X_s)$ a.s.

for all $s \in [t_1, t_2]$ <u>and such that $\|Z\|_\infty < \infty$ holds</u>.
In particular, we want all integrals to be well-defined and X, Y, Z to have values in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^{m \times d}$ respectively.
Furthermore, we call a function $u : (t, T] \times \mathbb{R}^n \to \mathbb{R}^m$ a Markovian decoupling field for $(\xi, (\mu, \sigma, f))$ on $(t, T]$ if u restricted to $[t', T]$ is a Markovian decoupling field for all $t' \in (t, T]$.

Note that a Markovian decoupling field is always a decoupling field in the standard sense as well. The only difference is that we are only interested in triples X, Y, Z, where Z is a.e.-bounded.
Regularity for Markovian decoupling fields is defined very similarly to standard regularity:

Definition 2.5.17. Let $u : [t, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ be a Markovian decoupling field to $(\xi, (\mu, \sigma, f))$. We call u *weakly regular*, if $L_{u,x} < L_{\sigma,z}^{-1}$ and $\sup_{s \in [t,T]} \|u(s, \cdot, 0)\|_\infty < \infty$.

Furthermore, we call such a u *strongly regular* if for all fixed $t_1, t_2 \in [t, T]$, $t_1 \le t_2$, the processes X, Y, Z arising in the defining property of a Markovian decoupling field are a.e. unique for each *constant* initial value $X_{t_1} = x \in \mathbb{R}^n$ and satisfy (2.27). In addition they must be measurable as functions of (x, s, ω) and even weakly differentiable w.r.t. $x \in \mathbb{R}^n$ such that for every $s \in [t_1, t_2]$ the mappings X_s and Y_s are measurable functions of (x, ω) and even weakly differentiable w.r.t. x such that (2.28) holds.

We say that a Markovian decoupling field u on $[t, T]$ is *strongly regular* on a subinterval $[t_1, t_2] \subseteq [t, T]$ if u restricted to $[t_1, t_2]$ is a strongly regular Markovian decoupling field for $(u(t_2, \cdot), (\mu, \sigma, f))$.

Furthermore, we say that a Markovian decoupling field $u : (t, T] \times \mathbb{R}^n \to \mathbb{R}^m$

- is weakly regular if u restricted to $[t', T]$ is weakly regular for all $t' \in (t, T]$,
- is strongly regular if u restricted to $[t', T]$ is strongly regular for all $t' \in (t, T]$.

Let us now come to the following local existence result for Markovian, but not necessarily Lipschitz problems:

Theorem 2.5.18. *Let*

- μ, σ, f *be*
  - *deterministic,*
  - *Lipschitz continuous in* x, y, z *on sets of the form* $[0, T] \times \mathbb{R}^n \times B_1 \times B_2$*, where* $B_1 \subset \mathbb{R}^m$ *and* $B_2 \subset \mathbb{R}^{m \times d}$ *are arbitrary bounded sets*
  - *and such that* $\|\mu(\cdot, 0, 0, 0)\|_\infty, \|f(\cdot, \cdot, 0, 0)\|_\infty, \|\sigma(\cdot, \cdot, 0, 0)\|_\infty, L_{\sigma,z} < \infty$,
- $\xi : \mathbb{R}^n \to \mathbb{R}^m$ *be bounded and such that* $L_{\xi,x} < L_{\sigma,z}^{-1}$.

Then there exists a time $t \in [0,T)$ such that $(\xi,(\mu,\sigma,f))$ has a unique bounded and weakly regular Markovian decoupling field u on $[t,T]$. This u is also

- *strongly regular,*
- *deterministic,*
- *continuous and*
- *satisfies $\sup_{t_1,t_2,X_{t_1}}(\|Y\|_\infty+\|Z\|_\infty)<\infty$, where $t_1\le t_2$ are from $[t,T]$ and X_{t_1} is an initial value (see the definition of a Markovian decoupling field for the meaning of these variables).*

Proof. For arbitrary constants $H,G>0$ let $\chi_H^G:\mathbb{R}^m\times\mathbb{R}^{m\times d}\to\mathbb{R}^m\times\mathbb{R}^{m\times d}$ be defined as

$$\chi_H^G(y,z):=\left(\mathbf{1}_{\{|y|<G\}}y+\frac{G}{|y|}\mathbf{1}_{\{|y|\ge G\}}y,\ \mathbf{1}_{\{|z|<H\}}z+\frac{H}{|z|}\mathbf{1}_{\{|z|\ge H\}}z\right).$$

It is straightforward to check that χ_H^G is Lipschitz continuous and bounded with 1 as Lipschitz constant. Also, $\chi_H^G(y,z)=(y,z)$ if $|y|\le G$ and $|z|\le H$.

We implement an "inner cutoff" by defining μ_H^G,σ_H^G,f_H^G via $\mu_H^G(t,x,y,z):=\mu(t,x,\chi_H^G(y,z))$, etc.

The boundedness of χ_H^G together with its Lipschitz continuity makes μ_H^G,σ_H^G,f_H^G Lipschitz continuous with some Lipschitz constant L_H^G. Furthermore, $L_{\sigma_H^G,z}\le L_{\sigma,z}$. According to Theorem 2.2.1 we know that the problem given by $(\xi,(\mu_H^G,\sigma_H^G,f_H^G))$ has a unique weakly regular decoupling field u on some small interval $[t',T]$, with $t'\in[0,T)$. We also know that this u is strongly regular according to Corollary 2.5.4. Furthermore, u is deterministic (Lemma 2.5.13) and continuous (Lemma 2.5.15). Note, however, that t' does depend on G,H.

We claim that for sufficiently large G,H and $t\in[t',T)$ this u will also be a Markovian decoupling field to $(\xi,(\mu,\sigma,f))$ on $[t,T]$:

Using Remark 2.2.3, we have

$$L_{u(t,\cdot),x}\le L_{\xi,x}+C_H^G(T-t)^{\frac14}\quad\forall t\in[t',T],$$

where $C_H^G<\infty$ is some constant, which does not depend on $t\in[t',T]$. We denote by $L_{\sigma,y}^G$ the Lipschitz constant of $\sigma_H^G(\cdot,\cdot,\cdot,0)$ w.r.t. $y\in\mathbb{R}^m$. Clearly this value does not depend on $H>0$. For any $t_1\in[t',T]$ and $\mathcal{F}_{t_1}$ - measurable initial value X_{t_1} consider the corresponding X,Y,Z on $[t_1,T]$ satisfying the forward equation, the backward equation and the decoupling condition for μ_H^G,σ_H^G,f_H^G and u. Note that σ_H^G is bounded for any $G,H>0$. So using Lemma 2.5.14 we have $\|Z\|_\infty<\infty$ and, therefore, also

$$\begin{aligned}\|Z\|_\infty&\le\frac{\sup_{s\in[t_1,T]}L_{u(s,\cdot),x}\cdot\left(\|\sigma(\cdot,\cdot,0,0)\|_\infty+L_{\sigma,y}^G\|Y\|_\infty\right)}{1-\sup_{s\in[t_1,T]}L_{u(s,\cdot),x}L_{\sigma,z}}\le\\&\le\frac{\left(L_{\xi,x}+C_H^G(T-t_1)^{\frac14}\right)\cdot\left(\|\sigma(\cdot,\cdot,0,0)\|_\infty+L_{\sigma,y}^G\|Y\|_\infty\right)}{1-L_{\xi,x}L_{\sigma,z}-L_{\sigma,z}C_H^G(T-t_1)^{\frac14}}=\\&=\frac{L_{\xi,x}\left(\|\sigma(\cdot,\cdot,0,0)\|_\infty+L_{\sigma,y}^G\|Y\|_\infty\right)}{1-L_{\xi,x}L_{\sigma,z}-L_{\sigma,z}C_H^G(T-t_1)^{\frac14}}+\frac{C_H^G(T-t_1)^{\frac14}\cdot\left(\|\sigma(\cdot,\cdot,0,0)\|_\infty+L_{\sigma,y}^G\|Y\|_\infty\right)}{1-L_{\xi,x}L_{\sigma,z}-L_{\sigma,z}C_H^G(T-t_1)^{\frac14}}\end{aligned}\tag{2.35}$$

for $T-t_1$ small enough.

We can bound u itself as well: Note that f_H^G is essentially bounded due to $\|f(\cdot,\cdot,0,0)\|_\infty < \infty$, the local Lipschitz continuity of f in Y, Z and the use of a cuttoff in Y and Z. Therefore the backward equation

$$Y_s + \int_s^T Z_r \,\mathrm{d}W_r = \xi(X_T) - \int_s^T f_H^G(r, X_r, Y_r, Z_r)\,\mathrm{d}r, \qquad s \in [t_1, T],$$

implies after applying conditional expectations

$$|Y_s| = \left| \mathbb{E}\left[\xi(X_T)|\mathcal{F}_s\right] - \mathbb{E}\left[\int_s^T f_H^G(r, X_r, Y_r, Z_r)\,\mathrm{d}r \middle| \mathcal{F}_s \right] \right| \leq \|\xi\|_\infty + (T - t_1)\tilde{C}_H^G, \tag{2.36}$$

for $s \in [t_1, T]$, where the constant $\tilde{C}_H^G \in (0, \infty)$ depends on G, H.

Now, we need to

- choose G large enough such that $\|\xi\|_\infty$ becomes smaller $\frac{G}{2}$ and then
- choose H large enough such that $\frac{L_{\xi,x}\left(\|\sigma(\cdot,\cdot,0,0)\|_\infty + L_{\sigma,y}^G \|Y\|_\infty\right)}{1 - L_{\xi,x}L_{\sigma,z}}$ becomes smaller $\frac{H}{4}$,

which determines $t' \in [0, T]$ as a function of H, G according to Remark 2.2.2. Then in the second step choose $t \in [t', T)$ close enough to T, such that

- $L_{\sigma,z}C_H^G(T-t)^{\frac{1}{4}}$ becomes smaller $\frac{1}{2}(1 - L_{\xi,x}L_{\sigma,z})$,
- $\frac{C_H^G(T-t)^{\frac{1}{4}}\left(\|\sigma(\cdot,\cdot,0,0)\|_\infty + L_{\sigma,y}^G \cdot G\right)}{1 - L_{\xi,x}L_{\sigma,z}}$ becomes smaller than $\frac{H}{4}$ and
- $\tilde{C}_H^G\sqrt{T-t}$ becomes smaller than $\frac{G}{2}$.

Considering (2.35) and (2.36) this implies that if $t_1 \in [t, T]$, the processes Y and Z a.e. do not leave the region in which the cutoff is "passive", i.e. the balls of radii G and H. Therefore, u restricted to the interval $[t, T]$ is a decoupling field to $(\xi, (\mu, \sigma, f))$, not just to $(\xi, (\mu_H^G, \sigma_H^G, f_H^G))$. It is even a Markovian decoupling field due to the (essential) boundedness of our Z. As a Markovian decoupling field it is weakly regular, because it is weakly regular as a decoupling field to $(\xi, (\mu_H^G, \sigma_H^G, f_H^G))$. Furthermore, u is bounded due to the decoupling condition and the boundedness of Y for arbitrary initial values X_{t_1}.

Uniqueness: Assume there is another bounded and weakly regular Markovian decoupling field $\tilde{u}$ to $(\xi, (\mu, \sigma, f))$on $[t, T]$. Choose a $t_1 \in [t, T]$ and an $x \in \mathbb{R}^n$ as initial condition $X_{t_1} = x$ and consider the corresponding processes $\tilde{X}, \tilde{Y}, \tilde{Z}$, which satisfy the corresponding FBSDE on $[t_1, T]$, together with the decoupling condition via $\tilde{u}$. The latter implies that $\tilde{Y}$ is bounded because $\tilde{u}$ is. At the same time consider X, Y, Z solving the same FBSDE on $[t_1, T]$, but associated with the Markovian decoupling field u. Since $\tilde{Y}, \tilde{Z}, Y, Z$ are bounded, the two triples $(\tilde{X}, \tilde{Y}, \tilde{Z})$ and (X, Y, Z) also solve some Lipschitz FBSDE given by $(\xi, (\mu_H^G, \sigma_H^G, f_H^G))$ on $[t_1, T]$ for $G, H > 0$ large enough. The two conditions $\tilde{Y}_s = \tilde{u}(s, \tilde{X}_s)$ and $Y_s = u(s, X_s)$ imply using Remark 2.2.4 that both triples are in $\mathbb{G}_0$ and coincide. In particular, $\tilde{u}(t_1, x) = \tilde{Y}_{t_1} = Y_{t_1} = u(t_1, x)$. ✓

Strong regularity of u as a Markovian decoupling field to $(\xi, (\mu, \sigma, f))$ follows directly from

- the above argument about uniqueness of X, Y, Z for deterministic initial values and bounded Z, where Y is bounded automatically due to boundedness of u,
- and the strong regularity of u as decoupling field to $(\xi, (\mu_H^G, \sigma_H^G, f_H^G))$.

□

Remark 2.5.19. We observe from the proof that the supremum of all $h = T - t$ with t satisfying the hypotheses of Theorem 2.5.18 can be bounded away from 0 by a bound, which depends only on

- $L_{\xi,x}$, $L_{\xi,x} \cdot L_{\sigma,z}$ and $\|\xi\|_\infty$,
- T, $\|f(\cdot,\cdot,0,0)\|_\infty$ and $\|\sigma(\cdot,\cdot,0,0)\|_\infty$,
- the values $(L^G_H)_{H\in[0,\infty)}$ where L^G_H is the Lipschitz constant of μ, σ and f on $[0,T]\times\mathbb{R}^n\times B^1_G\times B^2_H$ w.r.t. to the last 3 components, where $B^1_G \subset \mathbb{R}^m$ and $B^2_H \subset \mathbb{R}^{m\times d}$ are balls of radii G and H with center 0

and which is monotonically decreasing in these values.

The following natural concept introduces a type of Markovian decoupling field for non-Lipschitz problems (non-Lipschitz in z), to which nevertheless standard Lipschitz results can be applied.

Definition 2.5.20. Let u be a Markovian decoupling field for $(\xi,(\mu,\sigma,f))$. We call u *controlled in z* if there exists a constant $C > 0$ such that for all $t_1, t_2 \in [t,T]$, $t_1 \leq t_2$, and all initial values X_{t_1}, the corresponding processes X, Y, Z from the definition of a Markovian decoupling field satisfy $|Z_s(\omega)| \leq C$, for almost all $(s,\omega) \in [t_1,t_2]\times\Omega$. If for a fixed triplet (t_1,t_2,X_{t_1}) there are different choices for X, Y, Z, then all of them are supposed to satisfy the above control.

We say that a Markovian decoupling field u on $[t,T]$ is *controlled in z* on a subinterval $[t_1,t_2] \subseteq [t,T]$ if u restricted to $[t_1,t_2]$ is a Markovian decoupling field for $(u(t_2,\cdot),(\mu,\sigma,f))$ that is controlled in z.

Furthermore, we call a Markovian decoupling field on an interval $(s,T]$ *controlled in z* if it is controlled in z on every compact subinterval $[t,T] \subseteq (s,T]$ (with C possibly depending on t).

Remark 2.5.21. Our Markovian decoupling field u constructed in the proof Theorem 2.5.18 is obviously controlled in z: Consider (2.35) together with the choice of $t \leq t_1$ made in the proof.

Remark 2.5.22. Let μ, σ, f, ξ be as in Theorem 2.5.18 and assume that we have a Markovian decoupling field u on some interval $[t,T]$, which is weakly regular, bounded and controlled in z.
Then u is also a weakly regular decoupling field to a Lipschitz problem obtained through an inner cutoff as in the proof Theorem 2.5.18, where Y is bounded due to the decoupling condition. As such u is also strongly regular according to Corollary 2.5.4.
Furthermore, Lemma 2.5.13 is applicable since u is a weakly regular decoupling field to a Lipschitz problem. So, u is deterministic. But now Lemma 2.5.15 is also applicable, since due to the use of a cutoff we can assume the type of linear growth required there. So, u is also continuous.

Lemma 2.5.23. *Let μ, σ, f, ξ be as in Theorem 2.5.18. For $0 \leq s \leq t < T$ let u be a bounded and weakly regular Markovian decoupling field for $(\xi,(\mu,\sigma,f))$ on $[s,T]$.*
If u is controlled in z on $[s,t]$ and $T-t$ is small enough as required in Theorem 2.5.18 resp. Remark 2.5.19, then u is controlled in z on $[s,T]$.

Proof. Clearly, u is not just controlled in z on $[s,t]$, but also on $[t,T]$ (with a possibly different constant), according to Remark 2.5.21. Define C as the maximum of these two constants.
We only need to control Z by C for the case $s \leq t_1 \leq t \leq t_2 \leq T$, the other two cases being trivial.
Now, consider the processes X, Y, Z on the interval $[t_1,t_2]$ corresponding to some initial value X_{t_1} and fulfilling the forward equation, the backward equation and the decoupling condition. Since the restrictions of these processes to $[t_1,t]$ still fulfill these three properties we obtain $|Z_r(\omega)| \leq C$ for almost all $r \in [t_1,t], \omega \in \Omega$.
At the same time, if we restrict X, Y, Z to $[t,t_2]$, we observe that these restrictions satisfy the forward equation, the backward equation and the decoupling condition for the interval $[t,t_2]$ with X_t as initial value. Therefore, $|Z_r(\omega)| \leq C$ holds for a.a. $r \in [t,t_2], \omega \in \Omega$ as well. $\square$

As a consequence we can inductively show that sufficiently regular Markovian decoupling fields must be controlled in z.

Corollary 2.5.24. *Let μ, σ, f, ξ be as in Theorem 2.5.18. Assume that there exists a bounded and weakly regular Markovian decoupling field u to this problem on some interval $[t, T]$. Then u is controlled in z.*

Proof. Let $S \subseteq [t, T]$ be the set of all times $s \in [t, T]$, s.t. u is controlled in z on $[t, s]$.

- Clearly, $t \in S$: For the interval $[t, t] = \{t\}$ one can only choose $t_1 = t_2 = t$ and so $Z : [t, t] \times \Omega \to \mathbb{R}^{m \times d}$ is $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$ - a.e. 0, independently of the initial value X_{t_1}. So, we can take for C any positive value.

- Let $s \in S$ be arbitrary. According to Lemma 2.5.23 there exists an $h > 0$ s.t. u is controlled in z on $[t, (s+h) \wedge T]$, since $\|u((s+h) \wedge T, \cdot)\|_\infty < \infty$ and $L_{u((s+h) \wedge T, \cdot)} < L_{\sigma,z}^{-1}$. Considering Remark 2.5.19 and the requirements $\sup_{s \in [t,T]} \|u(s, \cdot)\|_\infty < \infty$, $L_{u,x} < L_{\sigma,z}^{-1}$, we can choose h independently of s.

This shows $S = [t, T]$ using small interval induction (forward). □

The property of Markovian decoupling fields to be controlled in z allows us to show the following two results as simple consequences of the theory developed in the Lipschitz case.

Corollary 2.5.25. *Let μ, σ, f, ξ be as in Theorem 2.5.18. Assume that there are two bounded and weakly regular Markovian decoupling fields $u^{(1)}, u^{(2)}$ to this problem on some interval $[t, T]$. Then $u^{(1)} = u^{(2)}$ up to modifications.*

Proof. We know that $u^{(1)}$ and $u^{(2)}$ are controlled in z according to Corollary 2.5.24. Choose a passive cutoff (see proof of Theorem 2.5.18) and apply Corollary 2.5.3. □

Corollary 2.5.26. *Let μ, σ, f, ξ be as in Theorem 2.5.18. Assume that there exists a bounded and weakly regular Markovian decoupling field u on some interval $[t, T]$. Then u is strongly regular.*

Proof. u is controlled in z according to Corollary 2.5.24. Choose a passive cutoff (see proof of Theorem 2.5.18) and apply Corollary 2.5.4. □

Remember the definition of the maximal interval. We aim at working with bounded Markovian decoupling fields. But in our current setting μ, σ, f may depend in a super-linear way on y, such that singularities may very well occur because of exploding u rather than exploding u_x. We, therefore, need to define a new type of maximal interval.

Definition 2.5.27. Let $I^b_{\max} \subseteq [0, T]$ for $(\xi, (\mu, \sigma, f))$ be the union of all intervals $[t, T] \subseteq [0, T]$ such that there exists a bounded and weakly regular Markovian decoupling field u on $[t, T]$.

We will sometimes refer to $I^b_{\max}$ as the *maximal interval*. However, it should not be confused with $I_{\max}$, which is defined in a different way (but fulfills a similar function).

Note that the above definition is only of interest if ξ is bounded.

Unfortunately the maximal interval $I^b_{\max}$ might very well be open to the left. Therefore, we need to make our notions more precise in the following definitions:

Definition 2.5.28. Let $t < T$.

- We call a function $u : (t, T] \times \mathbb{R}^n \to \mathbb{R}^m$ a Markovian decoupling field for $(\xi, (\mu, \sigma, f))$ on $(t, T]$ if u restricted to $[t', T]$ is a Markovian decoupling field for all $t' \in (t, T]$.

- We call a Markovian decoupling field u on $(t, T]$ *weakly regular* if u restricted to $[t', T]$ is a weakly regular Markovian decoupling field for all $t' \in (t, T]$.

- We call a Markovian decoupling field u on $(t,T]$ *strongly regular* if u restricted to $[t',T]$ is strongly regular for all $t'\in(t,T]$.

- We call a Markovian decoupling field u on $(t,T]$ *controlled in z* if u restricted to $[t',T]$ is controlled in z for all $t'\in(t,T]$.

According to the following theorem, we have existence and uniqueness on $I^b_{\max}$. In this result the condition on u to be locally bounded means that $\sup_{s\in[t,T]}\operatorname{ess\,sup}_{\omega\in\Omega}\sup_{x\in\mathbb{R}^n}|u(s,\omega,x)|<\infty$ for all $t\in I^b_{\max}$. If u is weakly regular, this is equivalent to $\sup_{s\in[t,T]}\|u(s,\cdot,\cdot)\|_\infty<\infty$. If u is a continuous function of time and space, this is equivalent to $\|u|_{[t,T]}\|_\infty<\infty$ for all $t\in I^b_{\max}$.

Theorem 2.5.29. *Let μ,σ,f,ξ be as in Theorem 2.5.18. Then there exists a unique locally bounded and weakly regular Markovian decoupling field u on $I^b_{\max}$. This u is also deterministic, continuous, controlled in z and strongly regular.*
Furthermore, either $I^b_{\max}=[0,T]$ or $I^b_{\max}=(t^b_{\min},T]$, where $0\le t^b_{\min}<T$.

Proof. Let $t\in I^b_{\max}$. According to the definition of $I^b_{\max}$ there exists a Markovian decoupling field $\check{u}^{(t)}$ on $[t,T]$ satisfying $L_{\check{u}^{(t)},x}<L^{-1}_{\sigma,z}$ and $\sup_{s\in[t,T]}\|\check{u}^{(t)}(s,\cdot)\|_\infty<\infty$. There is only one such $\check{u}^{(t)}$ according to Corollary 2.5.25. According to Corollary 2.5.24 and Remark 2.5.22 we can assume that $\check{u}^{(t)}$ is a continuous function on $[t,T]\times\mathbb{R}^n$. Furthermore, for $t,t'\in I^b_{\max}$ the functions $\check{u}^{(t)}$ and $\check{u}^{(t')}$ coincide on $[t\vee t',T]$ again because of Corollary 2.5.25.
Define $u(t,\cdot):=\check{u}^{(t)}(t,\cdot)$ for all $t\in I^b_{\max}$. This function u is a Markovian decoupling field on $[t,T]$, since it coincides with $\check{u}^{(t)}$ on $[t,T]$. Therefore, u is a Markovian decoupling field on the whole interval $I^b_{\max}$ and satisfies $L_{u|_{[t,T]},x}<L^{-1}_{\sigma,z}$, $\|u|_{[t,T]}\|_\infty<\infty$ for all $t\in I^b_{\max}$.
Uniqueness of u follows directly from Corollary 2.5.25 applied to every interval $[t,T]\subseteq I^b_{\max}$.
Furthermore, u is controlled in z and strongly regular on $[t,T]$ for all $t\in I^b_{\max}$ due to Corollaries 2.5.24 and 2.5.26.
Addressing the form of $I^b_{\max}$, we see that $I^b_{\max}=[t,T]$ with $t\in(0,T]$ is not possible. Assume otherwise. According to the above there exists a Markovian decoupling field u on $[t,T]$ s.t. $L_{u,x}<L^{-1}_{\sigma,z}$ and $\|u\|_\infty<\infty$. But then u can be extended a little bit to the left using Theorem 2.5.18 and Lemma 2.1.2. $\square$

The following result basically states that for a singularity to occur either u or u_x has to "explode" at $t^b_{\min}$.

Lemma 2.5.30. *Let μ,σ,f,ξ be as in Theorem 2.5.18. If $I^b_{\max}=(t^b_{\min},T]$, then*

$$\lim_{t\downarrow t_{\min}}\left(\left(1+L_{u(t,\cdot),x}\right)^{-1}-\left(1+L^{-1}_{\sigma,z}\right)^{-1}\right)\left(\|u(t,\cdot)\|_\infty+1\right)^{-1}=0, \tag{2.37}$$

where u is the Markovian decoupling field according to Theorem 2.5.29.

Proof. This can be shown by contradiction. Clearly,

$$L_{u(t,\cdot),x}<L^{-1}_{\sigma,z}\quad\Longleftrightarrow\quad\left(1+L_{u(t,\cdot),x}\right)^{-1}-\left(1+L^{-1}_{\sigma,z}\right)^{-1}>0$$

and

$$\|u(t,\cdot)\|_\infty<\infty\quad\Longleftrightarrow\quad\left(\|u(t,\cdot)\|_\infty+1\right)^{-1}>0$$

for all $t\in I^b_{\max}$. So, if (2.37) does not hold, we can select times $t_n\downarrow t^b_{\min}$ as $n\to\infty$ such that

$$\left(\left(1+L_{u(t_n,\cdot),x}\right)^{-1}-\left(1+L^{-1}_{\sigma,z}\right)^{-1}\right)\left(\|u(t_n,\cdot)\|_\infty+1\right)^{-1}$$

is bounded away from 0 uniformly in $n \in \mathbb{N}$. Since this value does not exceed the values

$$\left(\left(1 + L_{u(t_n,\cdot),x}\right)^{-1} - \left(1 + L_{\sigma,z}^{-1}\right)^{-1}\right) \text{ and } \left(\|u(t_n,\cdot)\|_\infty + 1\right)^{-1}$$

we obtain

$$\sup_{n\in\mathbb{N}} L_{u(t_n,\cdot),x} < L_{\sigma,z}^{-1} \quad \text{and} \quad \sup_{n\in\mathbb{N}} \|u(t_n,\cdot)\|_\infty < \infty.$$

Considering $u(t_n,\cdot)$ as a potential terminal condition for the time t_n, we choose an $h > 0$ according to Remark 2.5.19, such that it does not depend on n. This independence holds precisely because of the two inequalities above. Now, choose n large enough to have $t_n - t^b_{\min} < h$. So, u can be extended to the left using Theorem 2.5.18 and Lemma 2.1.2 to a weakly regular Markovian decoupling field on an interval $[(t_n - h) \vee 0, T]$, beyond the value $t^b_{\min}$, thereby contradicting the definition of $I^b_{\max}$. $\square$

Chapter 3

A Note on Regularizing Properties of Non-Degenerate Noise

In this chapter we will analyze coupled problems of the form

$$X_s = x + \int_0^s \mu(r, X_r, Y_r)\,\mathrm{d}r + \int_0^s \sigma(r)\,\mathrm{d}W_r,$$

$$Y_s = \xi(X_T) - \int_s^T f(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^T Z_r\,\mathrm{d}W_r, \quad s \in [0,T],$$

with real-valued X, Y and deterministic μ, σ, f, ξ where σ is *non-degenerate*. This means that we assume that $|\sigma|$, where σ is merely a measurable function of time, can be bounded away from 0. This property is crucial since otherwise the problem is ill-posed, i.e. cannot be solved on the whole interval $[0,T]$ in general.

Unlike the other chapters the goal of the current one is not to present truly new results. Instead, we highlight how decoupling fields can be used to show well-posedness, i.e. existence and uniqueness of solutions on the whole interval $[0,T]$, for coupled problems with non-degenerate σ using decoupling fields. In particular we do not utilize PDE theory as was done for instance in [Del02], where results from [LSU68] are used. As already indicated we make rather restrictive assumptions. It is our goal to merely indicate the technique and provide a purely stochastic interpretation of the regularizing influence of non-degenerate σ.
We proceed with summarizing the relevant results from Chapter 2. Using this theory from the previous chapter we show Theorem 3.1.16 which is an important technical result for problems satisfying $n = m = 1$. It will be utilized in adapted form in other chapters. The rest of the chapter is then devoted to the study of the particular class of problems introduced above. The technical preliminaries for this will be layed out in Section 3.2.1.

3.1 The method of decoupling fields

We consider families (μ, σ, f) of measurable functions, more precisely

$$\mu : [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \longrightarrow \mathbb{R}^n,$$

$$\sigma : [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \longrightarrow \mathbb{R}^{n\times d},$$

$$f : [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \longrightarrow \mathbb{R}^m,$$

where

- $n, m, d \in \mathbb{N}$ and $T > 0$,

- $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ is a complete filtered probability space,

- $\mathcal{F}_0 = \sigma(p) \vee \mathcal{N}$ for some $p : \Omega \to S$ and a polish space S, $\mathcal{N}$ are the null sets,

- $\mathcal{F}_t = \sigma(\mathcal{F}_0, (W_s)_{s\in[0,t]})$ holds, where $(W_t)_{t\in[0,T]}$ is a d-dimensional Brownian motion, independent of $\mathcal{F}_0$,

- $\mathcal{F} = \mathcal{F}_T$.

We want μ, σ and f to be progressively measurable w.r.t. $(\mathcal{F}_t)_{t\in[0,T]}$, i.e. $\mu\mathbf{1}_{[0,t]}, \sigma\mathbf{1}_{[0,t]}, f\mathbf{1}_{[0,t]}$ must be $\mathcal{B}([0,T]) \otimes \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^{m\times d})$ - measurable for all $t \in [0,T]$. We will assume throughout the chapter that μ, σ and f have this property without mentioning it.

Definition 3.1.1. Let $\xi : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ be measurable and let $t \in [0,T]$.
We call a function $u : [t,T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ with $u(T,\omega,\cdot) = \xi(\omega,\cdot)$ for a.a. $\omega \in \Omega$ a *decoupling field* for $(\xi, (\mu,\sigma,f))$ on $[t,T]$ if for all $t_1, t_2 \in [t,T]$ with $t_1 \le t_2$ and any $\mathcal{F}_{t_1}$ - measurable $X_{t_1} : \Omega \to \mathbb{R}^n$ there exist progressive processes X, Y, Z on $[t_1, t_2]$ such that

- $X_s = X_{t_1} + \int_{t_1}^s \mu(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_{t_1}^s \sigma(r, X_r, Y_r, Z_r)\,\mathrm{d}W_r$ a.s.,

- $Y_s = Y_{t_2} - \int_s^{t_2} f(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^{t_2} Z_r\,\mathrm{d}W_r$ a.s.,

- $Y_s = u(s, X_s)$ a.s.,

for all $s \in [t_1, t_2]$. In particular, we want all integrals to be well-defined and X, Y, Z to have values in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$ respectively.

Some remarks about this definition:

- The first of the above three equations is called the *forward equation*, the second the *backward equation* and the third will be referred to as the *decoupling condition*.

- This requirement that X should start at X_{t_1} is referred to as the *initial condition*. X_{t_1} is also sometimes referred to as the *initial value*.

- Note that if $t_2 = T$, we get $Y_T = \xi(X_T)$ a.s. as a consequence of the decoupling condition together with $u(T,\omega,\cdot) = \xi(\omega,\cdot)$ for a.a. $\omega \in \Omega$.

- If $t_2 = T$, we can say that a triple (X, Y, Z) solves the FBSDE, meaning that it satisfies the forward and the backward equation, together with $Y_T = \xi(X_T)$. This relationship $Y_T = \xi(X_T)$ is referred to as the *terminal condition*.

In the following will list a few important results regarding decoupling fields, the proofs of which can be found in Section 2.1.1:

In contrast to classical solutions of FBSDE, decoupling fields on different intervals can be pasted together.

Lemma 3.1.2. *Let u be a decoupling field for $(\xi, (\mu,\sigma,f))$ on $[t,T]$ and $\tilde{u}$ be a decoupling field for $(u(t,\cdot), (\mu,\sigma,f))$ on $[s,t]$, for $0 \le s < t < T$. Then, the map $\hat{u}$ given by $\hat{u} := \tilde{u}\mathbf{1}_{[s,t]} + u\mathbf{1}_{(t,T]}$ is a decoupling field for $(\xi, (\mu,\sigma,f))$ on $[s,T]$.*

We want to remark that if u is a decoupling field and $\tilde{u}$ is a modification of u, i.e. for each $s \in [t,T]$ the functions $u(s,\omega,\cdot)$ and $\tilde{u}(s,\omega,\cdot)$ coincide for almost all $\omega \in \Omega$, then $\tilde{u}$ is also a decoupling field to the same problem. So, u could also be referred to as a class of modifications. Some of the representatives of the class might be progressively measurable, others not. As we see below a progressively measurable representative does exist if the decoupling field is Lipschitz continuous in x:

Lemma 3.1.3. *Let $u\colon [t,T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ be a decoupling field to $(\xi,(\mu,\sigma,f))$ which is Lipschitz continuous in $x \in \mathbb{R}^n$ in the sense that there exists a constant $L > 0$ s.t. for every $s \in [t,T]$:*

$$|u(s,\omega,x) - u(s,\omega,x')| \leq L|x-x'| \qquad \forall x,x' \in \mathbb{R}^n, \quad \textit{for a.a. } \omega \in \Omega.$$

Then u has a modification $\tilde{u}$ which is progressively measurable.

Let $I \subseteq [0,T]$ be an interval and $u : I \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ a map such that $u(s,\cdot)$ is measurable for every $s \in I$. We define

$$L_{u,x} := \sup_{s\in I} \inf\{L \geq 0 \,|\, \text{for a.a. } \omega \in \Omega : |u(s,\omega,x) - u(s,\omega,x')| \leq L|x-x'| \text{ for all } x,x' \in \mathbb{R}^n\}.$$

where $\inf \emptyset := \infty$. We also set $L_{u,x} := \infty$ if $u(s,\cdot)$ is not measurable for every $s \in I$. One can easily show that $L_{u,x} < \infty$ is equivalent to u having a modification, which is truly Lipschitz continuous in $x \in \mathbb{R}^n$.

We denote by $L_{\sigma,z}$ the Lipschitz constant of σ w.r.t. its dependence on the last component z (and w.r.t. the Frobenius norms on $\mathbb{R}^{m\times d}$ and $\mathbb{R}^{n\times d}$). We set $L_{\sigma,z} = \infty$ if σ is not Lipschitz continuous in z.
By $L_{\sigma,z}^{-1} = \frac{1}{L_{\sigma,z}}$ we mean $\frac{1}{L_{\sigma,z}}$ if $L_{\sigma,z} > 0$ and ∞ otherwise.
Similarly one can define constants $L_{\mu,x}$, $L_{f,z}$, etc. When we refer to *the* Lipschitz constant we mean the infimum of all Lipschitz constants (which is again a Lipschitz constant).

Definition 3.1.4. Let $u\colon [t,T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ be a decoupling field to $(\xi,(\mu,\sigma,f))$. We say that u is *weakly regular* if $L_{u,x} < L_{\sigma,z}^{-1}$ and $\sup_{s\in[t,T]} \|u(s,\cdot,0)\|_\infty < \infty$.

This is a natural definition due to Lemma 3.1.3, in practice, however, it is important to have explicit knowledge about the regularity of (X,Y,Z). For instance, it is important to know in which spaces the processes live, and how they react to changes in the initial value. Specifically, it can be very useful to have differentiability of (X,Y,Z) w.r.t. the initial value.
In the following we need further notation. For an integrable real valued random variable F the expression $\mathbb{E}_t[F]$ refers to $\mathbb{E}[F|\mathcal{F}_t]$, while $\mathbb{E}_{\hat{t},\infty}[F]$ refers to $\operatorname{ess\,sup} \mathbb{E}[F|\mathcal{F}_t]$, which might be ∞, but is always well-defined as the infimum of all constants $c \in [-\infty,\infty]$ such that $\mathbb{E}[F|\mathcal{F}_t] \leq c$ a.s. Additionally, we write $\|F\|_\infty$ for the essential supremum of $|F|$, for an arbitrary measurable F.

Definition 3.1.5. Let $u\colon [t,T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ be a weakly regular decoupling field to $(\xi,(\mu,\sigma,f))$. We call u *strongly regular* if for all fixed $t_1,t_2 \in [t,T]$, $t_1 \leq t_2$, the processes (X,Y,Z) arising in the definition of a decoupling field are a.e unique and satisfy

$$\sup_{s\in[t_1,t_2]} \mathbb{E}_{t_1,\infty}[|X_s|^2] + \sup_{s\in[t_1,t_2]} \mathbb{E}_{t_1,\infty}[|Y_s|^2] + \mathbb{E}_{t_1,\infty}\left[\int_{t_1}^{t_2} |Z_s|^2 \,\mathrm{d}s\right] < \infty, \tag{3.1}$$

for each constant initial value $X_{t_1} = x \in \mathbb{R}^n$. In addition they must be measurable as functions of (x,s,ω) and even weakly differentiable w.r.t. $x \in \mathbb{R}^n$ such that for every $s \in [t_1,t_2]$ the mappings X_s and Y_s are measurable functions of (x,ω) and even weakly differentiable w.r.t. x such that

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \sup_{s\in[t_1,t_2]} \mathbb{E}_{t_1,\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}x}X_s\right|_v^2\right] < \infty,$$

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \sup_{s\in[t_1,t_2]} \mathbb{E}_{t_1,\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}x}Y_s\right|_v^2\right] < \infty,$$

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \mathbb{E}_{t_1,\infty}\left[\int_{t_1}^{t_2}\left|\frac{\mathrm{d}}{\mathrm{d}x}Z_s\right|_v^2 \mathrm{d}s\right] < \infty. \tag{3.2}$$

We say that a decoupling field u on $[t,T]$ is *strongly regular* on a subinterval $[t_1,t_2] \subseteq [t,T]$ if u restricted to $[t_1,t_2]$ is a strongly regular decoupling field for $(u(t_2,\cdot),(\mu,\sigma,f))$.

Strong regularity is a fundamental concept in our theory. It allows to work with weak derivatives and apply the rules of Lemmas A.2.4 to A.2.8 in particular. Consult Section 2.1.2 of Chapter 2 for more on the subject of weak derivatives.

Under certain conditions a rich existence, uniqueness and regularity theory for decoupling fields can be developed. We will summarize the main results, which are proven in Chapter 2, Section 2.5:

Definition 3.1.6. We say that ξ,μ,σ,f satisfy *standard Lipschitz conditions (SLC)* if

- μ,σ,f are Lipschitz continuous in (x,y,z) with Lipschitz constant L,
- $\|(|\mu|+|f|+|\sigma|)(\cdot,\cdot,0,0,0)\|_\infty < \infty$,
- $\xi:\Omega\times\mathbb{R}^n\to\mathbb{R}^m$ is measurable s.t. $\|\xi(\cdot,0)\|_\infty<\infty$ and $L_{\xi,x} < L_{\sigma,z}^{-1}$.

Theorem 3.1.7 (Global uniqueness)**.** *Let μ,σ,f,ξ satisfy SLC and assume that there are two weakly regular decoupling fields $u^{(1)},u^{(2)}$ to the corresponding problem on some interval $[t,T]$. Then $u^{(1)}=u^{(2)}$ up to modifications.*

Theorem 3.1.8 (Global regularity)**.** *Let μ,σ,f,ξ satisfy SLC and assume that there exists a weakly regular decoupling field u to this problem on some interval $[t,T]$.Then u is strongly regular.*

Notice that Theorem 3.1.7 only provides uniqueness of weakly regular decoupling fields, not uniqueness of processes (X,Y,Z) solving the FBSDE in the classical sense. However, using Theorem 3.1.8 one can show:

Corollary 3.1.9. *Let μ,σ,f,ξ satisfy SLC and assume that there exists a weakly regular decoupling field u on some interval $[t,T]$.*
Then for any initial condition $X_t=x\in\mathbb{R}^n$ there is a unique solution (X,Y,Z) of the FBSDE on $[t,T]$ satisfying

$$\sup_{s\in[t,T]}\mathbb{E}_{0,\infty}[|X_s|^2] + \sup_{s\in[t,T]}\mathbb{E}_{0,\infty}[|Y_s|^2] + \mathbb{E}_{0,\infty}\left[\int_t^T |Z_s|^2\,\mathrm{d}s\right] < \infty. \tag{3.3}$$

Now, we want to investigate how large the interval $[t,T]$ can be chosen, such that we still have (weakly regular) decoupling fields on this interval. It is natural to work with the following definition.

Definition 3.1.10. We define the *maximal interval* $I_{\max}\subseteq[0,T]$ for $(\xi,(\mu,\sigma,f))$ as the union of all intervals $[t,T]\subseteq[0,T]$, such that there exists a weakly regular decoupling field u on $[t,T]$.

Unfortunately the maximal interval might very well be open to the left. Therefore, we need to make our notions more precise in the following definitions.

Definition 3.1.11. Let $t<T$. We call a function $u:(t,T]\times\mathbb{R}^n\to\mathbb{R}^m$ a decoupling field for $(\xi,(\mu,\sigma,f))$ on $(t,T]$ if u restricted to $[t',T]$ is a decoupling field for all $t'\in(t,T]$.
We call a decoupling field u on $(t,T]$ *weakly regular* if u restricted to $[t',T]$ is weakly regular for all $t'\in(t,T]$. Similarly we call it *strongly regular* if u restricted to $[t',T]$ is strongly regular for all $t'\in(t,T]$.

Now, we can formulate

Theorem 3.1.12 (Global existence in weak form)**.** *Let μ, σ, f, ξ satisfy SLC. Then there exists a unique weakly regular decoupling field u on $I_{\max}$. This u is even strongly regular.*
Furthermore, either $I_{\max} = [0, T]$ or $I_{\max} = (t_{\min}, T]$ where $0 \leq t_{\min} < T$.

3.1.1 Some examples

Example 3.1.13. For $T \geq 1$ and a $t \in [0, T]$ consider the following FBSDE on the Interval $[t, T]$:

$$X_s = x + \int_t^s Y_r \,\mathrm{d}r,$$

$$Y_s = X_T - \int_s^T Z_r \,\mathrm{d}W_r, \quad s \in [t, T].$$

One can easily check that for $t > T - 1$ the associated SLC problem has the unique weakly regular decoupling field

$$u(s, x) = \frac{x}{1 - (T - s)}, \quad s \in [t, T], \quad x \in \mathbb{R},$$

such that the corresponding processes X, Y, Z on $[t, T]$ have the structure

$$X_s = x + (s - t)\frac{x}{1 - (T - t)} = x\frac{1 - (T - s)}{1 - (T - t)},$$

$$Y_s = \frac{x}{1 - (T - t)},$$

$$Z_s = 0,$$

such that $u(s, X_s) = Y_s$.
So, we can extend u to a weakly regular decoupling field on $(T - 1, T]$.
However, $u(s, x)$ goes to infinity for $s \downarrow (T - 1)$ and $x \neq 0$. At the same time $\lim_{s\downarrow(T-1)} L_{u(s,\cdot)} = \infty$, which indicates that the FBSDE might be considered ill-posed for $t = T - 1$.

Example 3.1.14. One could speculate that the problem of ill-posedness comes from non-boundedness of u. However, we can change the above problem, such that the terminal condition becomes bounded:

$$X_s = x + \int_t^s Y_r \,\mathrm{d}r,$$

$$Y_s = g(X_T) - \int_s^T Z_r \,\mathrm{d}W_r, \quad s \in [t, T],$$

where $g(x) := x$ for $x \in [-1, 1]$, $g(x) := 1$ for $x \in [1, \infty)$ and $g(x) := -1$ for $x \in (-\infty, -1]$.
For $t > T - 1$ the problem will have a unique weakly regular decoupling field

$$u(s, x) = \begin{cases} \frac{x}{1-(T-s)}, & \text{if } |x| \in [0, 1 - (T - s)] \\ 1, & \text{if } x \in [1 - (T - s), \infty) \\ -1, & \text{if } x \in (-\infty, -(1 - (T - s))] \end{cases}$$

where $s \in [t, T]$ and $x \in \mathbb{R}$. The corresponding processes X, Y, Z on $[t, T]$ have the form

$$X_s = x + (s - t)u(t, x),$$

$$Y_s = u(t, x),$$

$$Z_s = 0,$$

so

$$X_s = x + (s-t)u(t,x) = \begin{cases} x\frac{1-(T-s)}{1-(T-t)}, & \text{if } |x| \in [0, 1-(T-t)] \\ x + (s-t), & \text{if } x \in [1-(T-t), \infty) \\ x - (s-t), & \text{if } x \in (-\infty, -(1-(T-t))] \end{cases}$$

and, therefore, $u(s, X_s) = Y_s = u(t,x)$: An easy way to check this is to use that both functions $u(s, X_s)$ and $u(t,x)$ are continuous and piecewise linear in x having a weak derivative of

$$\frac{1}{1-(T-s)} \cdot \frac{1-(T-s)}{1-(T-t)} = 1-(T-t)$$

on the interval given by all $x \in \mathbb{R}$ satisfying

$$\left| x\frac{1-(T-s)}{1-(T-t)} \right| \leq 1-(T-s) \quad \Longleftrightarrow \quad |x| \leq \frac{1}{1-(T-t)}$$

and vanishing everywhere else. Also, $u(s, X_s)$ and $u(t,x)$ are both 0 for $x = 0$.
Note that even though $|u|$ is uniformly bounded by 1 the Lipschitz constant of $u(t,\cdot)$ converges to infinity as $t \downarrow T-1$!
Furthermore, the FBSDE becomes clearly ill-posed for $t = T-1$: Choosing $x = 0$ and $t = T-1$ we observe that the FBSDE

$$X_s = \int_{T-1}^{s} Y_r \, \mathrm{d}r,$$

$$Y_s = g(X_T) - \int_s^T Z_r \, \mathrm{d}W_r, \quad s \in [T-1, T],$$

has infinitely many solutions

$$X_s = (s-(T-1))y, \quad Y_s = y, \quad Z_s = 0, \quad s \in [T-1, T],$$

where y can be chosen arbitrarily from $[-1,1]$: Note that $X_T = y$ and so $g(X_T) = y$.

Later we will prove Theorem 3.2.1 which implies that the above problem becomes well-posed if we add some non-degenerate noise intensity $\sigma > 0$ to the forward equation s.t. the problem assumes the form

$$X_s = x + \int_t^s Y_r \, \mathrm{d}r + \int_t^s \sigma \, \mathrm{d}W_r,$$

$$Y_s = g(X_T) - \int_s^T Z_r \, \mathrm{d}W_r, \quad s \in [t, T].$$

In this case the problem can be uniquely solved on the whole interval $[0,T]$ regardless of how small $\sigma > 0$ is chosen! Also, Y, Z remain uniformly bounded.

3.1.2 Global existence in strong form

By *global existence in strong form* we mean the weak global existence of Theorem 3.1.12 together with $I_{\max} = [0,T]$. Unfortunately the "bad" case $I_{\max} = (t_{\min}, T]$ is possible and is even more common. The following result basically says that this case can only occur if there is an "explosion" in the spatial derivative of u as we approach the lower boundary $t_{\min}$. By "explosion" we mean reaching the "forbidden" value $L_{\sigma,z}^{-1}$ which is just ∞ in many applications.

Lemma 3.1.15. *Let μ, σ, f, ξ satisfy SLC. If $I_{\max} = (t_{\min}, T]$, then*

$$\lim_{t \downarrow t_{\min}} L_{u(t,\cdot),x} = L_{\sigma,z}^{-1},$$

where u is the unique decoupling field on $I_{\max}$.

Lemma 3.1.15 serves as a blueprint to show strong global existence in those cases in which it is suspected to hold. Let us describe the different steps.

1. Assume indirectly that $I_{\max} = [0, T]$ does not hold, which implies $I_{\max} = (t_{\min}, T]$. Choose arbitrary $t \in (t_{\min}, T]$, $x \in \mathbb{R}^n$ and consider the corresponding FBSDE.
2. Differentiate the FBSDE w.r.t. x. This is possible because of strong regularity of u (Theorem 3.1.12). We obtain joint dynamics of $\frac{\mathrm{d}}{\mathrm{d}x}X$, $\frac{\mathrm{d}}{\mathrm{d}x}Y$, $\frac{\mathrm{d}}{\mathrm{d}x}Z$.
3. Using Itô's formula deduce the dynamics of $\frac{\mathrm{d}}{\mathrm{d}x}Y_s(\frac{\mathrm{d}}{\mathrm{d}x}X_s)^{-1}$. Note that this process should be equal to $u_x(s, X_s)$, as a consequence of the decoupling condition $Y_s = u(s, X_s)$.
4. Using the dynamics of $u_x(s, X_s)$ show that its modulus can be bounded away from $L_{\sigma,z}^{-1}$ independently of t, x, s, ω. This contradicts Lemma 3.1.15 and, therefore, $I_{\max} = [0, T]$ must hold.

This blueprint can be referred to as the *method of decoupling fields* to show global existence of solutions to FBSDEs (note Corollary 3.1.9 at this point). Steps 1., 2. and 3. can be done in a rather general setting. Step 4., however, seems to be much more problem specific. Let us first conduct the third step for the following class of problems:

Theorem 3.1.16. *Let μ, σ, f, ξ satisfy SLC and assume furthermore $n = m = 1$. Choose any $[t, T] \subseteq I_{\max}$ and consider the corresponding X, Y, Z. If $\frac{\mathrm{d}}{\mathrm{d}x}X > 0$ a.e. for a.a. $x \in \mathbb{R}$, then $\hat{V}_s := \frac{\mathrm{d}}{\mathrm{d}x}u(s, X_s)$ has dynamics*

$$\begin{aligned}\hat{V}_s = \hat{V}_T - \int_s^T \hat{Z}_r \,\mathrm{d}W_r - \int_s^T \Bigg(&\delta_r^{f,x} + \delta_r^{f,y}\hat{V}_r + \delta_r^{f,z}\left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)^{-1}\left(\hat{Z}_r + \hat{V}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r\right)\right) - \\ &- \hat{V}_r\left(\delta_r^{\mu,x} + \delta_r^{\mu,y}\hat{V}_r + \delta_r^{\mu,z}\left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)^{-1}\left(\hat{Z}_r + \hat{V}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r\right)\right)\right) - \\ &- \left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)^{-1}\left(\hat{Z}_r + \hat{V}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r\right)\right)\right)\hat{Z}_r^\top \Bigg)\mathrm{d}r\end{aligned}$$

with

- *real-valued processes $\delta^{\mu,x}$, $\delta^{\mu,y}$, $\delta^{f,x}$, $\delta^{f,y}$ bounded by $L_{\mu,x}$, $L_{\mu,y}$, $L_{f,x}$, $L_{f,y}$ respectively,*
- *$\mathbb{R}^{1\times d}$ - valued processes $\delta^{\sigma,x}$, $\delta^{\sigma,y}$ bounded by $L_{\sigma,x}$, $L_{\sigma,y}$ respectively,*
- *$\mathbb{R}^{1\times(1\times d)}$ - valued processes $\delta^{\mu,z}$, $\delta^{f,z}$ bounded by $L_{\mu,z}$, $L_{f,z}$ and an*
- *$\mathbb{R}^{(1\times d)\times(1\times d)}$ - valued process $\delta^{\sigma,z}$ bounded by $L_{\sigma,z}$ (in the operator norm).*
- *$\hat{Z}$ is some progressive $\mathbb{R}^{1\times d}$ - valued process on $[t, T]$ s.t. $\int_t^T |\hat{Z}_r|^2 \,\mathrm{d}r < \infty$ a.s.*

The assumption $\frac{\mathrm{d}}{\mathrm{d}x}X > 0$ a.e. for a.a. $x \in \mathbb{R}$ will (for instance) hold if $L_{\mu,z} = L_{\sigma,z} = 0$.

Proof. Choose $t_1 \in [t, T]$ and $x \in \mathbb{R}$ and consider the corresponding X, Y, Z with

- $X_s = x + \int_{t_1}^s \mu(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_{t_1}^s \sigma(r, X_r, Y_r, Z_r)\,\mathrm{d}W_r$

- $Y_s = \xi(X_T) - \int_s^T f(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^T Z_r\,\mathrm{d}W_r$,

- $Y_s = u(s, X_s)$ a.s.,

where $s \in [t_1, T]$. For the following we will use strong regularity of the decoupling field:
We differentiate w.r.t. x using Lemma A.3.2:

$$\frac{\mathrm{d}}{\mathrm{d}x}X_s = 1 + \int_{t_1}^{s}\left(\delta_r^{\mu,x}\frac{\mathrm{d}}{\mathrm{d}x}X_r + \delta_r^{\mu,y}\frac{\mathrm{d}}{\mathrm{d}x}Y_r + \delta_r^{\mu,z}\frac{\mathrm{d}}{\mathrm{d}x}Z_r\right)\mathrm{d}r + \\ + \int_{t_1}^{s}\left(\delta^{\sigma,x}\frac{\mathrm{d}}{\mathrm{d}x}_r X_r + \delta_r^{\sigma,y}\frac{\mathrm{d}}{\mathrm{d}x}Y_r + \delta_r^{\sigma,z}\frac{\mathrm{d}}{\mathrm{d}x}Z_r\right)\mathrm{d}W_r$$

$$\frac{\mathrm{d}}{\mathrm{d}x}Y_s = \frac{\mathrm{d}}{\mathrm{d}x}Y_T - \int_s^T \frac{\mathrm{d}}{\mathrm{d}x}Z_r\,\mathrm{d}W_r - \int_s^T\left(\delta_r^{f,x}\frac{\mathrm{d}}{\mathrm{d}x}X_r + \delta_r^{f,y}\frac{\mathrm{d}}{\mathrm{d}x}_r Y_r + \delta_r^{f,z}\frac{\mathrm{d}}{\mathrm{d}x}_r Z_r\right)\mathrm{d}r,$$

for some processes $\delta^{\mu,x}$, $\delta^{\mu,y}$, $\delta^{\mu,z}$, $\delta^{\sigma,x}$, $\delta^{\sigma,y}$, $\delta^{\sigma,z}$, $\delta^{f,x}$, $\delta^{f,y}$, $\delta^{f,z}$ as required by the theorem.

To be precise the above equations hold a.s. for every $s \in [t_1, T]$, for almost all $x \in \mathbb{R}$. So, the now following implications of the above hold for almost every $x \in \mathbb{R}$.

Write $U_r := \frac{\mathrm{d}}{\mathrm{d}x}X_r$, $V_r := \frac{\mathrm{d}}{\mathrm{d}x}Y_r$ and $\tilde{Z}_r := \frac{\mathrm{d}}{\mathrm{d}x}Z_r$ for short. Note that U and V are $\mathbb{R}$ - valued and $\tilde{Z}$ is $\mathbb{R}^{1\times d}$ - valued. We can also assume without loss of generality that U and V are continuous in time. So, we have a linear FBSDE

$$U_s = 1 + \int_{t_1}^{s}\left(\delta_r^{\mu,x}U_r + \delta_r^{\mu,y}V_r + \delta_r^{\mu,z}\tilde{Z}_r\right)\mathrm{d}r + \int_{t_1}^{s}\left(\delta_r^{\sigma,x}U_r + \delta_r^{\sigma,y}V_r + \delta_r^{\sigma,z}\tilde{Z}_r\right)\mathrm{d}W_r$$

$$V_s = V_T - \int_s^T \tilde{Z}_r\,\mathrm{d}W_r - \int_s^T\left(\delta_r^{f,x}U_r + \delta_r^{f,y}V_r + \delta_r^{f,z}\tilde{Z}_r\right)\mathrm{d}r.$$

Also, defining $\hat{U} := U^{-1}$ we can rewrite

$$U_s = 1 + \int_{t_1}^{s}\left(\delta_r^{\mu,x} + \delta_r^{\mu,y}\hat{V}_r + \delta_r^{\mu,z}\tilde{Z}_r\hat{U}_r\right)U_r\,\mathrm{d}r + \int_{t_1}^{s}\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r\right)U_r\,\mathrm{d}W_r.$$

This means that if $L_{\mu,z} = L_{\sigma,z} = 0$, then U has linear dynamics and, therefore, remains positive on the whole $[t_1, T]$. Otherwise, we have to assume $U > 0$ a.e.
Define the bounded process $\hat{V}$ via $\hat{V}_r := \frac{\mathrm{d}}{\mathrm{d}x}u(r, X_r)$. Note that we can assume without loss of generality that $\left|\frac{\mathrm{d}}{\mathrm{d}x}u(r,\cdot)\right|$ is uniformly bounded by $L_{u(r,\cdot),x}$. Using the chain rule of Lemma A.3.1 we have $V_r = \hat{V}_r U_r$. Remembering $\hat{U} = U^{-1} = \frac{1}{U}$ we have $\hat{V} = V\hat{U}$. In particular, $\hat{V}$ has a modification which is continuous in time.
Applying the Itô formula to $\hat{U} = U^{-1}$ we obtain

$$\hat{U}_s = 1 - \int_{t_1}^{s}\hat{U}_r\Bigg(\delta_r^{\mu,x} + \delta_r^{\mu,y}\hat{V}_r + \delta_r^{\mu,z}\tilde{Z}_r\hat{U}_r - \\ - \left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r\right)\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r\right)^{\top}\Bigg)\mathrm{d}r - \\ - \int_{t_1}^{s}\hat{U}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r\right)\mathrm{d}W_r.$$

Applying the Itô formula to $\hat{V} = V\hat{U}$ we get

$$\hat{V}_s = \hat{V}_T - \int_s^T \tilde{Z}_r\hat{U}_r - \hat{V}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r\right)\mathrm{d}W_r - \\ - \int_s^T \Bigg\{\delta_r^{f,x} + \delta_r^{f,y}\hat{V}_r + \delta_r^{f,z}\tilde{Z}_r\hat{U}_r - \\ - \hat{V}_r\left(\delta_r^{\mu,x} + \delta_r^{\mu,y}\hat{V}_r + \delta_r^{\mu,z}\tilde{Z}_r\hat{U}_r - \left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r\right)\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r\right)^\top\right) - \\ - \left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r\right)\tilde{Z}_r^\top\hat{U}_r\Bigg\}\,\mathrm{d}r.$$

In the above equation we can effectively merge the marked terms by defining a process $\hat{Z}$ via

$$\hat{Z}_r := \tilde{Z}_r\hat{U}_r - \hat{V}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r\right),$$

so we get

$$\hat{V}_s = \hat{V}_T - \int_s^T \hat{Z}_r\,\mathrm{d}W_r - \int_s^T \Bigg(\delta_r^{f,x} + \delta_r^{f,y}\hat{V}_r + \delta_r^{f,z}\,\tilde{Z}_r\hat{U}_r - \\ - \hat{V}_r\left(\delta_r^{\mu,x} + \delta_r^{\mu,y}\hat{V}_r + \delta_r^{\mu,z}\,\tilde{Z}_r\hat{U}_r\right) - \left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\,\tilde{Z}_r\hat{U}_r\right)\hat{Z}_r^\top\Bigg)\,\mathrm{d}r. \quad (3.4)$$

An easy calculation starting from the definition of $\hat{Z}_r$ yields

$$\hat{Z}_r + \hat{V}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r\right) = \left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)\tilde{Z}_r\hat{U}_r,$$

$$\tilde{Z}_r\hat{U}_r = \left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)^{-1}\left(\hat{Z}_r + \hat{V}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r\right)\right), \quad (3.5)$$

where $I_d \in \mathbb{R}^{(1\times d)\times(1\times d)}$ is the identity. Note here that $\left\|\hat{V}\right\|_\infty \leq L_{u|[t_1,T]},x} < L_{\sigma,z}^{-1}$ and also that the operator norm of $\delta^{\sigma,z}$ is universally bounded by $L_{\sigma,z}$, so the essential supremum of the operator norm of $\hat{V}_r\delta_r^{\sigma,z}$ is strictly smaller than 1 and, therefore, the expression $\left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)^{-1}$ is well-defined and even universally bounded (on $[t_1, T]$) in the operator norm.
By plugging (3.5) into (3.4) we obtain:

$$\hat{V}_s = \hat{V}_T - \int_s^T \hat{Z}_r\,\mathrm{d}W_r - \int_s^T \Bigg\{\delta_r^{f,x} + \delta_r^{f,y}\hat{V}_r + \delta_r^{f,z}\left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)^{-1}\left(\hat{Z}_r + \hat{V}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r\right)\right) - \\ - \hat{V}_r\left(\delta_r^{\mu,x} + \delta_r^{\mu,y}\hat{V}_r + \delta_r^{\mu,z}\left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)^{-1}\left(\hat{Z}_r + \hat{V}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r\right)\right)\right) - \\ - \left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)^{-1}\left(\hat{Z}_r + \hat{V}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r\right)\right)\right)\hat{Z}_r^\top\Bigg\}\,\mathrm{d}r.$$

□

3.1.3 Markovian case

A problem given by μ, σ, f, ξ is said to be *Markovian*, if these four functions are deterministic, i.e. depend on t, x, y, z only.
In the Markovian case we can somewhat relax the Lipschitz continuity assumption and still obtain local existence together with uniqueness. What makes the Markovian case so special is the property

$$"Z_s = u_x(s, X_s) \cdot \sigma(s, X_s, Y_s, Z_s)"$$

which comes from the fact that u will also be deterministic. This property allows us to bound Z by a constant if we assume that σ is bounded.

In the following we will sum up results for the Markovian case which are proven in Section 2.5.1.

Lemma 3.1.17. *Let μ, σ, f, ξ satisfy SLC and assume in addition that they are deterministic. Assume that we have a weakly regular decoupling field u on an interval $[t, T]$. Then u is deterministic in the sense that it has a modification which is a function of $(r, x) \in [t, T] \times \mathbb{R}^n$ only.*

An application of Lemma 3.1.17 is the following quite fundamental result.

Lemma 3.1.18. *Let $(\xi, (\mu, \sigma, f))$ satisfy SLC and suppose that they are deterministic. Let u be a weakly regular decoupling field on an interval $[t, T]$. Choose $t_1 < t_2$ from $[t, T]$ and an initial condition X_{t_1}. Then the corresponding Z satisfies $\|Z\|_\infty \leq L_{u,x} \cdot \|\sigma\|_\infty$.*

This prospect for boundedness of Z in the Markovian case motivates the following definition, which will allow us to develop a theory for non-Lipschitz problems:

Definition 3.1.19. Let $\xi : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ be measurable and let $t \in [0, T]$.
We call a function $u : [t, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ with $u(T, \omega, \cdot) = \xi(\omega, \cdot)$ for a.a. $\omega \in \Omega$ a *Markovian decoupling field* for $(\xi, (\mu, \sigma, f))$ on $[t, T]$ if for all $t_1, t_2 \in [t, T]$ with $t_1 \leq t_2$ and any $\mathcal{F}_{t_1}$ - measurable $X_{t_1} : \Omega \to \mathbb{R}^n$ there exist progressive processes X, Y, Z on $[t_1, t_2]$ such that

- $X_s = X_{t_1} + \int_{t_1}^s \mu(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_{t_1}^s \sigma(r, X_r, Y_r, Z_r)\,\mathrm{d}W_r$ a.s.,
- $Y_s = Y_{t_2} - \int_s^{t_2} f(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^{t_2} Z_r\,\mathrm{d}W_r$ a.s.,
- $Y_s = u(s, X_s)$ a.s.

for all $s \in [t_1, t_2]$ and such that $\|Z\|_\infty < \infty$ holds.
In particular, we want all integrals to be well-defined and X, Y, Z to have values in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$ respectively.
Furthermore, we call a function $u : (t, T] \times \mathbb{R}^n \to \mathbb{R}^m$ a Markovian decoupling field for $(\xi, (\mu, \sigma, f))$ on $(t, T]$ if u restricted to $[t', T]$ is a Markovian decoupling field for all $t' \in (t, T]$.

Note that a Markovian decoupling field is always a decoupling field in the standard sense as well. The only difference is that we are only interested in triples (X, Y, Z), where Z is a.e. bounded.
Regularity for Markovian decoupling fields is defined very similarly to standard regularity:

Definition 3.1.20. Let $u : [t, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ be a Markovian decoupling field to $(\xi, (\mu, \sigma, f))$.
We call u *weakly regular*, if $L_{u,x} < L_{\sigma,z}^{-1}$ and $\sup_{s\in[t,T]} \|u(s, \cdot, 0)\|_\infty < \infty$.

We call a weakly regular u *strongly regular* if for all fixed $t_1, t_2 \in [t, T]$, $t_1 \leq t_2$, the processes (X, Y, Z) arising in the defining property of a Markovian decoupling field are a.e. unique for each *constant* initial value $X_{t_1} = x \in \mathbb{R}^n$ and satisfy (3.1). In addition they must be measurable as functions of (x, s, ω) and even weakly differentiable w.r.t. $x \in \mathbb{R}^n$ such that for every $s \in [t_1, t_2]$ the mappings X_s and Y_s are measurable functions of (x, ω) and even weakly differentiable w.r.t. x such that (3.2) holds.

We say that a Markovian decoupling field u on $[t,T]$ is *strongly regular* on a subinterval $[t_1,t_2] \subseteq [t,T]$ if u restricted to $[t_1,t_2]$ is a strongly regular Markovian decoupling field for $(u(t_2,\cdot),(\mu,\sigma,f))$.

Furthermore, we say that a Markovian decoupling field $u : (t,T] \times \mathbb{R}^n \to \mathbb{R}^m$

- is weakly regular if u restricted to $[t',T]$ is weakly regular for all $t' \in (t,T]$,
- is strongly regular if u restricted to $[t',T]$ is strongly regular for all $t' \in (t,T]$.

Now, we define a class of problems for which an existence and uniqueness theory will be developed:

Definition 3.1.21. We say that ξ,μ,σ,f satisfy *standard local Lipschitz conditions (SLLC)* if

- μ,σ,f are
  - deterministic,
  - Lipschitz continuous in x,y,z on sets of the form $[0,T] \times \mathbb{R}^n \times B_1 \times B_2$, where $B_1 \subset \mathbb{R}^m$ and $B_2 \subset \mathbb{R}^{m\times d}$ are arbitrary bounded sets
  - and such that $\|\mu(\cdot,0,0,0)\|_\infty, \|f(\cdot,\cdot,0,0)\|_\infty, \|\sigma(\cdot,\cdot,0,0)\|_\infty, L_{\sigma,z} < \infty$,
- $\xi : \mathbb{R}^n \to \mathbb{R}^m$ is bounded and such that $L_{\xi,x} < L_{\sigma,z}^{-1}$.

The following natural concept introduces a type of Markovian decoupling field for non-Lipschitz problems (non-Lipschitz in z), to which nevertheless standard Lipschitz results can be applied.

Definition 3.1.22. Let u be a Markovian decoupling field for $(\xi,(\mu,\sigma,f))$. We call u *controlled in z* if there exists a constant $C > 0$ such that for all $t_1,t_2 \in [t,T]$, $t_1 \le t_2$, and all initial values $X_{t_1} = x \in \mathbb{R}^n$, the corresponding processes X,Y,Z from the definition of a Markovian decoupling field satisfy $|Z_s(\omega)| \le C$, for almost all $(s,\omega) \in [t,T] \times \Omega$. If for a fixed triple (t_1,t_2,X_{t_1}) there are different choices for X,Y,Z, then all of them are supposed to satisfy the above control.

We say that a Markovian decoupling field u on $[t,T]$ is *controlled in z* on a subinterval $[t_1,t_2] \subseteq [t,T]$ if u restricted to $[t_1,t_2]$ is a Markovian decoupling field for $(u(t_2,\cdot),(\mu,\sigma,f))$ that is controlled in z.

Furthermore, we call a Markovian decoupling field on an interval $(s,T]$ *controlled in z* if it is controlled in z on every compact subinterval $[t,T] \subseteq (s,T]$ (with C possibly depending on t).

The following important result allows us to connect the SLLC - case to SLC.

Theorem 3.1.23. *Let μ,σ,f,ξ satisfy SLLC and assume that there exists a bounded and weakly regular Markovian decoupling field u to this problem on some interval $[t,T]$. Then u is controlled in z.*

Note at this point that if u is bounded, $Y = u(\cdot,X)$ will be bounded as well and so u will be a decoupling field to an SLC problem if we cutoff μ,σ,f appropriately. We can, thereby, extend the whole theory to SLLC problems:

Theorem 3.1.24 (Global uniqueness)**.** *Let μ,σ,f,ξ satisfy SLLC and assume that there are two bounded and weakly regular Markovian decoupling fields $u^{(1)},u^{(2)}$ to this problem on some interval $[t,T]$. Then $u^{(1)} = u^{(2)}$.*

Theorem 3.1.25 (Global regularity)**.** *Let μ,σ,f,ξ satisfy SLLC and assume that there exists a bounded and weakly regular Markovian decoupling field u to the corresponding FBSDE on some interval $[t,T]$. Then u is strongly regular.*

Definition 3.1.26. Let $I^b_{\max} \subseteq [0,T]$ for $(\xi,(\mu,\sigma,f))$ be the union of all intervals $[t,T] \subseteq [0,T]$ such that there exists a bounded and weakly regular decoupling field u on $[t,T]$.

Theorem 3.1.27 (Global existence in weak form)**.** *Let μ, σ, f, ξ satisfy SLLC. Then there exists a unique locally bounded and weakly regular Markovian decoupling field u on $I^b_{\max}$. This u is also controlled in z and strongly regular.*
Furthermore, either $I^b_{\max} = [0,T]$ or $I^b_{\max} = (t^b_{\min}, T]$, where $0 \le t^b_{\min} < T$.

The following result basically states that for a singularity to occur either u or u_x has to "explode" at $t_{\min}$.

Lemma 3.1.28. *Let μ, σ, f, ξ satisfy SLLC. If $I^b_{\max} = (t^b_{\min}, T]$, then*

$$\lim_{t \downarrow t^b_{\min}} \left(\left(1 + L_{u(t,\cdot),x}\right)^{-1} - \left(1 + L^{-1}_{\sigma,z}\right)^{-1} \right) \left(\|u(t,\cdot)\|_\infty + 1\right)^{-1} = 0,$$

where u is the Markovian decoupling field according to Theorem 3.1.27.

Similar to Lemma 3.1.15 the above result serves as a blueprint to show strong global existence for SLLC problems in those cases in which it is suspected to hold. Let us describe the different steps:

1. Using the dynamics of Y deduce a uniform bound for u.
2. Assume indirectly that $I^b_{\max} = [0,T]$ does not hold, which implies $I^b_{\max} = (t^b_{\min}, T]$. Choose arbitrary $t \in (t^b_{\min}, T]$, $x \in \mathbb{R}^n$ and consider the corresponding FBSDE.
3. Differentiate the FBSDE w.r.t. x. This is possible because of strong regularity of u (Theorem 3.1.25). We obtain joint dynamics of $\frac{\mathrm{d}}{\mathrm{d}x}X$, $\frac{\mathrm{d}}{\mathrm{d}x}Y$, $\frac{\mathrm{d}}{\mathrm{d}x}Z$.
4. Using Itô's formula deduce the dynamics of $\frac{\mathrm{d}}{\mathrm{d}x}Y_s(\frac{\mathrm{d}}{\mathrm{d}x}X_s)^{-1}$. Note that this process should be equal to $u_x(s, X_s)$, as a consequence of the decoupling condition $Y_s = u(s, X_s)$.
5. Using the dynamics of $u_x(s, X_s)$ show that its modulus can be bounded away from $L^{-1}_{\sigma,z}$ independently of t, x, s, ω. This contradicts Lemma 3.1.15 and, therefore, $I^b_{\max} = [0,T]$ must hold.

3.2 Main result

Our goal is to prove the following statement:

Theorem 3.2.1. *Let μ, σ, f, ξ satisfy SLLC and assume in addition:*

- $1 = n = m$,
- $0 = L_{\mu,z} = L_{\sigma,x} = L_{\sigma,y} = L_{\sigma,z}$,
- $|\sigma|$ *is uniformly bounded away from* 0,
- $\|f\|_\infty + L_{f,z} + \|\mu(\cdot,\cdot,0,0)\|_\infty < \infty$.

Then $I^b_{\max} = [0,T]$, i.e. strong global existence holds.

According to the method of decoupling fields for SLLC problems, we need to control u and u_x (see Lemma 3.1.28). Controlling u is easy:

Lemma 3.2.2. *Let μ, σ, f, ξ satisfy SLLC and assume furthermore that f is bounded. Then the decoupling field u on $I^b_{\max}$ is bounded by $\|\xi\|_\infty + T\|f\|_\infty$.*

Proof. For any $t_1, t_2 \in I^b_{\max}$, $t_1 \le t_2$ and $x \in \mathbb{R}$ consider the corresponding X, Y, Z with

- $X_s = x + \int_{t_1}^s \mu(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_{t_1}^s \sigma(r, X_r, Y_r, Z_r)\,\mathrm{d}W_r$,
- $Y_s = Y_{t_2} - \int_s^{t_2} f(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^{t_2} Z_r\,\mathrm{d}W_r$,
- $Y_s = u(s, X_s)$, $s \in [t_1, t_2]$.

Now, choose $t_2 = T$ and set $s = t_1$ and apply conditional expectations on both sides of the backward equation:

$$u(t_1, x) = Y_{t_1} = \mathbb{E}[\xi(X_T)|\mathcal{F}_{t_1}] - \int_{t_1}^T \mathbb{E}[f(r, X_r, Y_r, Z_r)|\mathcal{F}_{t_1}]\,\mathrm{d}r$$

and so

$$|u(t_1, x)| \leq \|\xi\|_\infty + T\|f\|_\infty.$$

Note $u(t_1, x) = u(t_1, X_{t_1}) = Y_{t_1}$. Here $t_1 \in I^b_{\max}$ and $x \in \mathbb{R}^n$ are arbitrary. $\square$

Controlling u_x is more challenging. It will be based on Theorem 3.1.16 and surprisingly Lemma 3.2.2, which means that the boundedness of u will play an important role in bounding its derivative. However, we will have to lay some additional groundwork before coming to the proof of the main result.

3.2.1 Some helpful lemmas

For the following lemma define $\chi(x) := 2F\left(x^{-1}\right) - 1$ for $x > 0$ and $\chi(0) := 1$, where F is the cumulative distribution function of the standard normal distribution. In particular, $\chi : \mathbb{R}^+ \to [0, 1]$ is continuous, monotonically decreasing and converges to zero for $x \to \infty$.
Let also ρ with $\rho(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, $x \in \mathbb{R}$ be the density of the standard normal distribution.

Lemma 3.2.3. *Let $h : \mathbb{R} \to \mathbb{R}$ be some measurable function s.t. $\|h\|_\infty := \operatorname{ess\,sup}_{x \in \mathbb{R}} |h(x)| < \infty$ and*

$$C_h := \sup_{a,b \in \mathbb{R}} \left| \int_a^b h(y)\,\mathrm{d}y \right| < \infty.$$

For $\sigma \in (0, \infty)$ consider $\tilde{h} : \mathbb{R} \to \mathbb{R}$ given by $\tilde{h}(x) := \int_{\mathbb{R}} h(x + y\sigma)\rho(y)\,\mathrm{d}y$. Then

$$\left\|\tilde{h}\right\|_\infty \leq \|h\|_\infty \cdot \chi\left(\frac{\sigma\|h\|_\infty}{C_h}\right).$$

Proof. In case $h = 0$ a.e. the statement is obvious, thus we can assume $\|h\|_\infty, C_h > 0$.
We want to prove

$$\left| \int_{\mathbb{R}} h(x + y\sigma)\rho(y)\,\mathrm{d}y \right| \leq \|h\|_\infty \cdot \chi\left(\frac{\sigma\|h\|_\infty}{C_h}\right), \quad x \in \mathbb{R}.$$

Since $\|h\|$ and C_h do not change under translations of h, we can assume without loss of generality that $x = 0$ and only consider the value $\int_{\mathbb{R}} h(y\sigma)\rho(y)\,\mathrm{d}y = \int_{\mathbb{R}} h(y)\rho_\sigma(y)\,\mathrm{d}y$, where $\rho_\sigma(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}$.

Define $G(x) := \int_0^x h(y)\,\mathrm{d}y$ for all $x \in \mathbb{R}$. We have using integration by parts:

$$\int_0^r h(y)\rho_\sigma(y)\,\mathrm{d}y = G(r)\rho_\sigma(r) - \int_0^r G(y)\rho'_\sigma(y)\,\mathrm{d}y$$

for all $r \geq 0$. We know that G is uniformly bounded by C_h. We also know that $\rho_\sigma(r)$ converges to 0 for $r \to \infty$. Thus

$$\int_0^\infty h(y)\rho_\sigma(y)\,\mathrm{d}y = \lim_{r \to \infty} \int_0^r h(y)\rho_\sigma(y)\,\mathrm{d}y = -\int_0^\infty G(x)\rho'_\sigma(x)\,\mathrm{d}x.$$

Now, define $R := \frac{C_h}{\|h\|_\infty}$.
For $0 \leq x \leq R$ the value $G(x)$ is bounded by $x \cdot \|h\|_\infty$. For $x \geq R$ the value $G(x)$ remains bounded by C_h. Keeping in mind that $\rho'_\sigma(x) \leq 0$ for all $x \geq 0$ we have

$$\int_0^\infty h(y)\rho_\sigma(y)\,\mathrm{d}y \leq -\int_0^R x \cdot \|h\|_\infty \rho'_\sigma(x)\,\mathrm{d}x - \int_R^\infty C_h \rho'_\sigma(x)\,\mathrm{d}x =$$

$$= -R\|h\|_\infty \rho_\sigma(R) + \int_0^R \|h\|_\infty \rho_\sigma(x)\,\mathrm{d}x - \int_R^\infty C_h \rho'_\sigma(x)\,\mathrm{d}x =$$

$$= -R\|h\|_\infty \rho_\sigma(R) + \int_0^R \|h\|_\infty \rho_\sigma(x)\,\mathrm{d}x + C_h \rho_\sigma(R) = \int_0^R \|h\|_\infty \rho_\sigma(x)\,\mathrm{d}x,$$

where we used integration by parts and $R\|h\|_\infty = C_h$. Finally we have

$$\int_0^\infty h(y)\rho_\sigma(y)\,\mathrm{d}y \leq \int_0^R \|h\|_\infty \rho_\sigma(x)\,\mathrm{d}x = \|h\|_\infty \int_0^{R/\sigma} \rho(y)\,\mathrm{d}y =$$

$$= \|h\|_\infty \left(F\left(\frac{R}{\sigma}\right) - F(0) \right) = \|h\|_\infty \left(F\left(\frac{C_h}{\sigma\|h\|_\infty}\right) - F(0) \right) = \frac{1}{2}\|h\|_\infty \cdot \chi\left(\frac{\sigma\|h\|_\infty}{C_h}\right).$$

Similarly we can also show

$$\int_0^\infty -h(y)\rho_\sigma(y)\,\mathrm{d}y \leq \frac{1}{2}\|h\|_\infty \cdot \chi\left(\frac{\sigma\|h\|_\infty}{C_h}\right),$$

thus obtaining

$$\left|\int_0^\infty h(y)\rho_\sigma(y)\,\mathrm{d}y\right| \leq \frac{1}{2}\|h\|_\infty \cdot \chi\left(\frac{\sigma\|h\|_\infty}{C_h}\right).$$

In the same manner we can prove

$$\left|\int_{-\infty}^0 h(y)\rho_\sigma(y)\,\mathrm{d}y\right| \leq \frac{1}{2}\|h\|_\infty \cdot \chi\left(\frac{\sigma\|h\|_\infty}{C_h}\right).$$

This shows using triangle inequality:

$$\left|\int_{\mathbb{R}} h(y)\rho_\sigma(y)\,\mathrm{d}y\right| \leq \|h\|_\infty \cdot \chi\left(\frac{\sigma\|h\|_\infty}{C_h}\right).$$

$\square$

It is known that super-linear, especially quadratic, ODEs can explode in finite time. However, we can prove:

Lemma 3.2.4. *Assume $\chi : [0,\infty) \to [0,1]$ is any function which is monotonically decreasing and converges to some value $\chi_{\min} \in [0,1)$.*
Let $\varphi : (t,T] \to [0,\infty)$ be some continuous function, which satisfies

$$\varphi_s \leq \varphi_r \cdot \chi(\varphi_r\sqrt{r-s}) + \int_s^r C(1+\varphi_q^p)\,\mathrm{d}q$$

for all $s, r \in (t,T]$, $s \leq r$, where $C > 0$, $p \in (1,3)$ are some constants. Then $\liminf_{s\downarrow t} \varphi(s) < \infty$.

Proof. We assume $\lim_{s\downarrow t}\sup_{q\in[s,T]}\varphi_q = \limsup_{s\downarrow t}\varphi_s = \infty$. Otherwise nothing needs to be proven. This assumption implies $\lim_{s\downarrow t}\sup_{q\in[s,r]}\varphi_q = \infty$ for all $r\in(t,T]$.
Let $\delta\in(1,\infty)$ be some constant we will specify later.
For any fixed $r\in(t,T]$ with $\varphi_r>0$ there exists an $s\in(t,r)$ such that

$$r-s=\frac{1}{\delta\cdot C\cdot\left(\sup_{q\in[s,r]}\varphi_q^{p-1}+\left(\sup_{q\in[s,r]}\varphi_q\right)^{-1}\right)}=\frac{\sup_{q\in[s,r]}\varphi_q}{\delta C\left(1+\sup_{q\in[s,r]}\varphi_q^p\right)}. \tag{3.6}$$

This is because the continuous function

$$s\longmapsto\frac{1}{\delta C\left(\sup_{q\in[s,r]}\varphi_q^{p-1}+\left(\sup_{q\in[s,r]}\varphi_q\right)^{-1}\right)}-(r-s)$$

has a positive value for $s=r$ and then converges to $-(r-t)<0$ for $s\downarrow t$.
Equation (3.6) implies

$$\int_s^r C(1+\varphi_q^p)\,\mathrm{d}q\le(r-s)C\left(1+\sup_{q\in[s,r]}\varphi_q^p\right)=\frac{1}{\delta}\sup_{q\in[s,r]}\varphi_q, \tag{3.7}$$

Now, observe $\varphi_r\chi(\varphi_r\sqrt{r-s})\le\varphi_r$ and therefore

$$\sup_{q\in[s,r]}\varphi_q\le\varphi_r+\int_s^r C(1+\varphi_q^p)\,\mathrm{d}q\le\varphi_r+\frac{1}{\delta}\sup_{q\in[s,r]}\varphi_q,$$

due to (3.7). A simple transformation implies

$$\sup_{q\in[s,r]}\varphi_q\le\left(1-\frac{1}{\delta}\right)^{-1}\varphi_r. \tag{3.8}$$

Combined with (3.7) we obtain:

$$\begin{aligned}\varphi_s\le\varphi_r\cdot\chi(\varphi_r\sqrt{r-s})+\int_s^r C(1+\varphi_q^p)\,\mathrm{d}q\le\\ \le\varphi_r\chi(\varphi_r\sqrt{r-s})+\frac{1}{\delta}\sup_{q\in[s,r]}\varphi_q\le\varphi_r\chi(\varphi_r\sqrt{r-s})+\frac{1}{\delta}\left(1-\frac{1}{\delta}\right)^{-1}\varphi_r.\end{aligned}$$

Now, choose $\delta:=\frac{3-\chi_{\min}}{1-\chi_{\min}}$, such that $\frac{1}{\delta}\left(1-\frac{1}{\delta}\right)^{-1}=\frac{1-\chi_{\min}}{3-\chi_{\min}}\left(\frac{3-\chi_{\min}-1+\chi_{\min}}{3-\chi_{\min}}\right)^{-1}=\frac{1-\chi_{\min}}{2}$ and so using (3.6) and (3.8)

$$\varphi_s\le\varphi_r\chi(\varphi_r\sqrt{r-s})+\frac{1-\chi_{\min}}{2}\varphi_r=$$

$$=\varphi_r\chi\left(\frac{\varphi_r\sqrt{\sup_{q\in[s,r]}\varphi_q}}{\sqrt{\delta C}\sqrt{1+\sup_{q\in[s,r]}\varphi_q^p}}\right)+\frac{1-\chi_{\min}}{2}\varphi_r\le\varphi_r\chi\left(\frac{\varphi_r\sqrt{\varphi_r}}{\sqrt{\delta C}\sqrt{1+\left(1-\frac{1}{\delta}\right)^{-p}\varphi_r^p}}\right)+\frac{1-\chi_{\min}}{2}\varphi_r,$$

where we used monotonicity of χ. Now, note

$$\left(1-\frac{1}{\delta}\right)^{-p}=\left(\frac{3-\chi_{\min}-1+\chi_{\min}}{3-\chi_{\min}}\right)^{-p}=\left(\frac{3-\chi_{\min}}{2}\right)^p\le\left(\frac{3}{2}\right)^3=\frac{27}{8}\le 4,$$

so if we assume $\varphi_r \geq 1$, then

$$\frac{\sqrt{\varphi_r^p}}{\sqrt{1+\left(1-\frac{1}{\delta}\right)^{-p}\varphi_r^p}} \geq \frac{\sqrt{\varphi_r^p}}{\sqrt{5\varphi_r^p}} \geq \frac{1}{3}$$

and, therefore,

$$\varphi_s \leq \varphi_r \chi\left(\frac{\sqrt{\varphi_r^{3-p}}}{\sqrt{\delta C}}\cdot\frac{1}{3}\right)+\frac{1-\chi_{\min}}{2}\varphi_r \leq \varphi_r\left(\frac{1-\chi_{\min}}{2}+\chi(\kappa)\right),$$

for all $\kappa > 0$, s.t. $\varphi_r \geq \left(9\delta C\kappa^2\right)^{\frac{1}{3-p}} \iff \frac{(\varphi_r)^{3-p}}{9\delta C} \geq \kappa^2 \iff \frac{\sqrt{\varphi_r^{3-p}}}{3\sqrt{\delta C}} \geq \kappa$.

Now, choose any $\kappa > 0$ such that $\chi(\kappa) < \frac{1+\chi_{\min}}{2}$ and $\tilde{\kappa} := \left(9\delta C\kappa^2\right)^{\frac{1}{3-p}} > 1$. Such a $\kappa > 0$ exists, since $\lim_{\kappa\to\infty}\chi(\kappa) = \chi_{\min}$ and $C, \delta, \frac{1}{3-p} > 0$. We now claim that $\liminf_{s\downarrow t}\varphi(s) \leq \tilde{\kappa}$:
For any $r \in (t,T]$ such that $\varphi_r \geq \tilde{\kappa} > 1 > 0$ choose an $s \in (t,r)$ according to the above procedure. Then $\varphi_s \leq \varphi_r\gamma$ with $\gamma := \left(\frac{1-\chi_{\min}}{2}+\chi(\kappa)\right) < 1$ will hold. If $\varphi_s \geq \tilde{\kappa}$, we can treat s as the new r and obtain an $s' \in (t,s)$ s.t. $\varphi_{s'} \leq \varphi_s\gamma \leq \varphi_r\gamma^2$. We repeat this procedure as long as $\varphi_{s^{(k)}} \geq \tilde{\kappa}$, $k = 1,2,3,\ldots$. In each step $\varphi_{s^{(k)}} \leq \varphi_r\gamma^{k+1}$. Since $\varphi_r < \infty$, for k large enough, $\varphi_{s^{(k)}} < \tilde{\kappa}$ will finally hold and the procedure will terminate. This means that there exists an $\hat{s} \in (t,r)$ s.t. $\varphi_{\hat{s}} < \tilde{\kappa}$.
To sum up: If we choose a sequence $(r_i)_{i\in\mathbb{N}}$ in $(t,T]$ with $\lim_{i\to\infty} r_i = t$ and $\varphi_{r_i} \geq \tilde{\kappa}$ for all i, we also get a sequence $(s_i)_{i\in\mathbb{N}}$, $s_i \in (t,r_i)$ with $\lim_{i\to\infty} s_i = t$ and $\varphi_{s_i} < \tilde{\kappa}$ for all i. This shows $\liminf_{s\downarrow t}\varphi(s) \leq \tilde{\kappa}$. $\square$

In order to apply the above lemma the following result will be useful in practice. For this particular result to hold the filtration $(\mathcal{F}_s)_{s\in[t,T]}$ does not have to be generated by the Brownian motion $(W_s)_{s\in[t,T]}$.

Lemma 3.2.5. *Let Y be some real-valued progressive bounded process on an interval $[t,T]$ with dynamics*

$$Y_s = Y_r - \int_s^r \left(a_q + b_qY_q + c_q|Y_q|^\alpha + Z_qd_q + |Y_q|^\gamma Z_qe_q\right)\mathrm{d}q - \int_s^r Z_q\,\mathrm{d}W_q, \qquad s,r \in [t,T],\ s<r,$$

where

- *a, b, c are bounded real-valued progressive processes,*
- *d, e are bounded $\mathbb{R}^d$ - valued progressive processes,*
- *Z is a progressive $\mathbb{R}^{1\times d}$ - valued process s.t. $\mathbb{E}\left[\int_t^T |Z_s|^2\,\mathrm{d}s\right] < \infty$ and*
- *$\alpha \in [1,3)$, $\gamma \in [0,\frac{1}{2})$ are some constants.*

Assume furthermore that there exists a measurable and bounded function $u : [t,T]\times\mathbb{R}\to\mathbb{R}$, such that $Y_s = u(s,X_s)$ a.s. for all $s \in [t,T]$, where X is an adapted Gaussian process on $[t,T]$ with $X_s - X_r$ being independent of $\mathcal{F}_r$ for all $r \in [t,T]$, $s \in [r,T]$. Assume also

$$\sup_{s\in[t,T]}\sup_{a,b\in\mathbb{R}}\left|\int_a^b u(s,x)\,\mathrm{d}x\right| \leq C_u < \infty,$$

$$\inf_{s,r\in[t,T]}\frac{\mathrm{Var}(X_s-X_r)}{|s-r|} \geq \sigma_0^2 > 0,$$

for some constants C_u and σ_0. Then $\varphi_s := \|Y_s\|_\infty + (1+\|Y_s\|_\infty^p)^{\frac{1}{p}} - 1$, where $p := \frac{3}{2} - \gamma$, satisfies

$$\varphi_s \le \varphi_r \cdot \chi(\varphi_r \sqrt{r-s}) + \int_s^r C(1+\varphi_q^\delta)\,\mathrm{d}q, \qquad s,r \in [t,T],\ s<r,$$

for some function χ which satisfies the properties from Lemma 3.2.4 and some constants $C>0$, $\delta \in (1,3)$. Furthermore, we can choose χ, C, δ such that

- *χ only depends on C_u, σ_0 and T,*
- *C only depends on $\|a\|_\infty$, $\|b\|_\infty$, $\|c\|_\infty$, $\|d\|_\infty$, $\|e\|_\infty$ and is monotonically increasing in these values,*
- *δ depends only on α and γ and is monotonically increasing in these values.*

Proof. For $p = \frac{3}{2} - \gamma \in (1, \frac{3}{2}]$ define a function $g : \mathbb{R} \to \mathbb{R}$ via

$$g(y) := (1+|y|^p)^{\frac{1}{p}} - 1, \qquad y \in \mathbb{R}.$$

Note that

- g is twice weakly differentiable s.t.
- $g'(y) = \operatorname{sgn}(y)|y|^{p-1}(1+|y|^p)^{\frac{1}{p}-1}$ and
- $g''(y) = (p-1)|y|^{p-2}(1+|y|^p)^{\frac{1}{p}-1} + |y|^{p-1}p|y|^{p-1}\left(\frac{1}{p}-1\right)(1+|y|^p)^{\frac{1}{p}-2}$, which can be simplified to
$$\begin{aligned} g''(y) &= (p-1)|y|^{p-2}(1+|y|^p)^{\frac{1}{p}-2}(1+|y|^p) + |y|^{p-2}|y|^p\,(1-p)\,(1+|y|^p)^{\frac{1}{p}-2} = \\ &= (p-1)|y|^{p-2}(1+|y|^p)^{\frac{1}{p}-2}\,(1+|y|^p-|y|^p) = (p-1)|y|^{p-2}(1+|y|^p)^{\frac{1}{p}-2}, \quad y \in \mathbb{R}\backslash\{0\}. \end{aligned}$$
Note that $|g''|$ is bounded by $|y|^{p-2}$, where $p-2 \in (-1, -\frac{1}{2}]$, which makes it locally integrable.
- $g(0) = 0$, which is also the minimum of g.
- g is convex due to $g'' \ge 0$.
- $g(y) \le |y|$ due to $(1+|y|^p)^{\frac{1}{p}} \le 1+|y| \iff 1+|y|^p \le (1+|y|)^p$, which is true since $p \ge 1$.
- $|y| \le g(y)+1$ due to $|y| \le (1+|y|^p)^{\frac{1}{p}} \iff |y|^p \le 1+|y|^p$.

Now, define $V_s := (1+|Y_s|^p)^{\frac{1}{p}} - 1 = g(Y_s)$ and apply the Itô formula to it:

$$V_s = V_r - \int_s^r \hat{V}_q\,\mathrm{d}q - \int_s^r \tilde{V}_q\,\mathrm{d}W_q, \qquad s,r \in [t,T],\ s<r,$$

where

$$\begin{aligned} \hat{V}_s := \operatorname{sgn}(Y_s)|Y_s|^{p-1}\,(1+|Y_s|^p)^{\frac{1}{p}-1}\,(a_s + b_sY_s + c_s|Y_s|^\alpha + Z_sd_s + |Y_s|^\gamma Z_s e_s) + \\ + \frac{1}{2}(p-1)|Y_s|^{p-2}\,(1+|Y_s|^p)^{\frac{1}{p}-2}\,Z_s^2 \end{aligned}$$

and

$$\tilde{V}_s := \operatorname{sgn}(Y_s)|Y_s|^{p-1}\,(1+|Y_s|^p)^{\frac{1}{p}-1}\,Z_s.$$

Since Z is in L^2, the process $\tilde{V}$ is in L^2 as well, due to

$$|Y_s|^{p-1}\left(1+|Y_s|^p\right)^{\frac{1}{p}-1} = \left(\frac{|Y_s|}{\left(1+|Y_s|^p\right)^{\frac{1}{p}}}\right)^{p-1} \leq 1. \tag{3.9}$$

So, we can apply conditional expectations:

$$V_s = \mathbb{E}[V_r|\mathcal{F}_s] - \mathbb{E}\left[\int_s^r \hat{V}_q\,\mathrm{d}q\middle|\mathcal{F}_s\right] \leq (1+\|Y_r\|_\infty^p)^{\frac{1}{p}} - 1 - \int_s^r \mathbb{E}\left[\hat{V}_q\middle|\mathcal{F}_s\right]\,\mathrm{d}q. \tag{3.10}$$

At the same time

$$Y_s = \mathbb{E}\left[u(r, X_s + (X_r - X_s))\middle|\mathcal{F}_s\right] - \int_s^r \mathbb{E}\left[a_q + b_q Y_q + c_q|Y_q|^\alpha + Z_q d_q + |Y_q|^\gamma Z_q e_q\middle|\mathcal{F}_s\right]\,\mathrm{d}q$$

and, so, according to Lemma 3.2.3:

$$\begin{aligned}
|Y_s| &\leq \left|\mathbb{E}\left[u(r, X_s + (X_r - X_s))\middle|\mathcal{F}_s\right]\right| + \int_s^r \mathbb{E}\left[\left|a_q + b_q Y_q + c_q|Y_q|^\alpha + Z_q d_q + |Y_q|^\gamma Z_q e_q\right|\middle|\mathcal{F}_s\right]\,\mathrm{d}q \leq \\
&\leq \left|\int_{\mathbb{R}} u\left(r, X_s + \mathbb{E}[X_r - X_s] + \sqrt{\mathrm{Var}(X_r - X_s)}y\right)\rho(y)\,\mathrm{d}y\right| + \\
&\qquad + \int_s^r \mathbb{E}\left[\left|a_q + b_q Y_q + c_q|Y_q|^\alpha + Z_q d_q + |Y_q|^\gamma Z_q e_q\right|\middle|\mathcal{F}_s\right]\,\mathrm{d}q \leq \\
&\leq \|u(r,\cdot)\|_\infty \cdot \check{\chi}\left(\frac{\sigma_0\sqrt{r-s}\|u(r,\cdot)\|_\infty}{C_u}\right) + \int_s^r \mathbb{E}\left[C_1(1+|Y_q|^\alpha) + C_2(1+|Y_q|^\gamma)|Z_q|\middle|\mathcal{F}_s\right]\,\mathrm{d}q,
\end{aligned} \tag{3.11}$$

where

- ρ is the density of the standard normal distribution,
- $\check{\chi}(x) := 2F\left(x^{-1}\right) - 1$ for $x > 0$ and $\check{\chi}(0) := 1$,
- $\sqrt{\mathrm{Var}(X_r - X_s)} \geq \sigma_0\sqrt{r-s}$,
- $\|u(r,\cdot)\|_\infty = \|Y_r\|_\infty$,
- $C_1, C_2 > 0$ are some constants which depend only on $\|a\|_\infty$, $\|b\|_\infty$, $\|c\|_\infty$, $\|d\|_\infty$, $\|e\|_\infty$ and are monotonically increasing in these values.

Now, we sum up the inequalities (3.10) and (3.11) to obtain

$$\begin{aligned}
|Y_s| + V_s &\leq \left(\|Y_r\|_\infty \cdot \check{\chi}\left(\frac{\sigma_0\sqrt{r-s}\|Y_r\|_\infty}{C_u}\right) + (1+\|Y_r\|_\infty^p)^{\frac{1}{p}} - 1\right) + \\
&\quad + \int_s^r \mathbb{E}\left[C_1'(1+|Y_q|^\alpha) + C_2'(1+|Y_q|^\gamma)|Z_q| - \frac{1}{2}(p-1)|Y_q|^{p-2}\left(1+|Y_q|^p\right)^{\frac{1}{p}-2}Z_q^2\middle|\mathcal{F}_s\right]\,\mathrm{d}q,
\end{aligned} \tag{3.12}$$

where

- we used the structure of $\hat{V}$ while keeping (3.9) in mind,
- $C_1', C_2' > 0$ are some constants which depend only on $\|a\|_\infty$, $\|b\|_\infty$, $\|c\|_\infty$, $\|d\|_\infty$, $\|e\|_\infty$ and are monotonically increasing in these values.

Observe

$$\frac{\|Y_r\|_\infty \cdot \check{\chi}\left(\frac{\sigma_0\sqrt{r-s}\|Y_r\|_\infty}{C_u}\right) + (1+\|Y_r\|_\infty^p)^{\frac{1}{p}} - 1}{\|Y_r\|_\infty + (1+\|Y_r\|_\infty^p)^{\frac{1}{p}} - 1} \le \frac{\check{\chi}\left(\frac{\sigma_0\sqrt{r-s}\|Y_r\|_\infty}{C_u}\right) + 1}{2}, \tag{3.13}$$

which is a consequence of the inequality $\lambda x + (1-\lambda)y \le \frac{1}{2}x + \frac{1}{2}y$ for $0 \le x \le y$, $\lambda \in [\frac{1}{2}, 1]$.
Since $\|Y_r\|_\infty \ge \frac{1}{2}(\|Y_r\|_\infty + g(\|Y_r\|_\infty)) = \frac{1}{2}\varphi_r$ we have

$$\check{\chi}\left(\frac{\sigma_0\sqrt{r-s}\|Y_r\|_\infty}{C_u}\right) \le \check{\chi}\left(\frac{\sigma_0\sqrt{r-s}\varphi_r}{2C_u}\right).$$

Now, we can finally define

$$\chi(x) := \frac{1}{2}\left(\check{\chi}\left(\frac{\sigma_0 x}{2C_u}\right) + 1\right), \qquad x \ge 0.$$

χ is continuous, starts at 0, decreases monotonically and converges to $\frac{1}{2}$.
So, using (3.13) we can rewrite (3.12):

$$\begin{aligned} |Y_s| + V_s \le \varphi_r \cdot \chi(\varphi_r\sqrt{r-s}) + \\ + \int_s^r \mathbb{E}\left[C_1'(1+|\varphi_q|^\alpha) + C_2'(1+|Y_q|^\gamma)|Z_q| - \frac{1}{2}(p-1)|Y_q|^{p-2}(1+|Y_q|^p)^{\frac{1}{p}-2} Z_q^2 \Big| \mathcal{F}_s\right] \mathrm{d}q. \end{aligned}$$

It remains to control the term $C_2'(1+|Y_q|^\gamma)Z_q$:

$$\begin{aligned} & C_2'(1+|Y_q|^\gamma)Z_q \le \\ & \le \frac{1}{2}(C_2')^2(1+|Y_q|^\gamma)^2\left((p-1)|Y_q|^{p-2}(1+|Y_q|^p)^{\frac{1}{p}-2}\right)^{-1} + \frac{1}{2}(p-1)|Y_q|^{p-2}(1+|Y_q|^p)^{\frac{1}{p}-2} Z_q^2 = \\ & = C_3'(1+|Y_q|^\gamma)^2|Y_q|^{2-p}(1+|Y_q|^p)^{2-\frac{1}{p}} + \frac{1}{2}(p-1)|Y_q|^{p-2}(1+|Y_q|^p)^{\frac{1}{p}-2} Z_q^2 \le \\ & \le C_4'\left(1+|Y_q|^{2\gamma+2-p+2p-1}\right) + \frac{1}{2}(p-1)|Y_q|^{p-2}(1+|Y_q|^p)^{\frac{1}{p}-2} Z_q^2, \end{aligned}$$

where $C_3', C_4' > 0$ are some constants which depend only on C_2' and are monotonically increasing in this value.
Now, observe $2\gamma + 2 - p + 2p - 1 = 2\gamma + p + 1 = \gamma + \frac{3}{2} + 1 \in [\frac{5}{2}, 3)$.
So, we end up with

$$\begin{aligned} |Y_s| + V_s \le \varphi_r \cdot \chi(\varphi_r\sqrt{r-s}) + \int_s^r C_1'(1+|\varphi_q|^\alpha) + C_4'(1+|\varphi_q|^{2\gamma+p+1})\,\mathrm{d}q \le \\ \le \varphi_r \cdot \chi(\varphi_r\sqrt{r-s}) + \int_s^r C(1+|\varphi_q|^\delta)\,\mathrm{d}q, \end{aligned}$$

where

- δ is the maximum of α and $2\gamma + p + 1$,
- C is some constant which depends only on $C_1', C_4' > 0$ and is monotonically increasing in these values.

It remains to remark that

$$\||Y_s| + V_s\|_\infty = \||Y_s| + g(Y_s)\|_\infty = \|Y_s\|_\infty + g(\|Y_s\|_\infty) = \varphi_s.$$

□

3.2.2 Proof of the main result

It remains to put the pieces together:

Proof of Theorem 3.2.1. Assume indirectly that $I^b_{\max}$ is not the whole interval, which means that it has the form $I^b_{\max} = \left(t^b_{\min}, T\right]$, for some $t^b_{\min} \in [0,T)$. Let u be the unique locally bounded and weakly regular Markovian decoupling field from Theorem 3.1.27. According to Lemma 3.2.2 u is bounded by $\|\xi\|_\infty + T\|f\|_\infty$. Since $L^{-1}_{\sigma,z} = \infty$ we have $\left(1 + L^{-1}_{\sigma,z}\right)^{-1} = 0$. According to Lemma 3.1.28 and Lemma 2.1.4

$$\lim_{t \downarrow t^b_{\min}} \|u_x(t,\cdot)\|_\infty = \infty \tag{3.14}$$

must hold. Now, choose $t \in I^b_{\max}$, $x \in \mathbb{R}$ and consider the corresponding X, Y, Z on $[t,T]$. Note that Y and Z are bounded and so u can be interpreted as a solution to an SLC problem if we use an appropriate cutoff. Theorem 3.1.16 is, therefore, applicable. We use from now on the notations from this result. Note that $\delta^{\mu,z}$, $\delta^{\sigma,x}$, $\delta^{\sigma,y}$ and $\delta^{\sigma,z}$ vanish:

$$\hat{V}_s = \hat{V}_T - \int_s^T \hat{Z}_r \,\mathrm{d}W_r - \int_s^T \left(\delta^{f,x}_r + \delta^{f,y}_r \hat{V}_r + \delta^{f,z}_r \hat{Z}_r - \hat{V}_r\left(\delta^{\mu,x}_r + \delta^{\mu,y}_r \hat{V}_r\right)\right) \mathrm{d}r, \quad s \in [t,T],$$

where $\hat{V}_s$ was defined as $u_x(s,X_s) = \frac{\mathrm{d}}{\mathrm{d}x} Y_s \left(\frac{\mathrm{d}}{\mathrm{d}x} X_s\right)^{-1}$.
Note that $\hat{Z}$ is in $BMO(\mathbb{P})$ according to Theorem A.1.11.
Furthermore, X satisfies dynamics

$$X_s = x + \int_t^s \mu(r, X_r, Y_r)\,\mathrm{d}r + \int_t^s \sigma(r)\,\mathrm{d}W_r = x + \int_t^s \sigma(r)\,\mathrm{d}\tilde{W}_r, \quad s \in [t,T],$$

where

$$\tilde{W}_s := W_s + \int_t^s \mu(r, X_r, Y_r) \frac{\sigma(r)^\top}{|\sigma(r)|^2}\,\mathrm{d}r, \quad s \in [t,T]$$

defines a Brownian motion under some probability measure $\tilde{\mathbb{P}}$ obtained from an appropriate Girsanov measure change. Note that

$$\gamma_r := \mu(r, X_r, Y_r) \frac{\sigma(r)^\top}{|\sigma(r)|^2}$$

is bounded by some universal constant due to $\|\mu(\cdot,\cdot,0)\|_\infty < \infty$ and $\|Y\|_\infty \leq \|\xi\|_\infty + T\|f\|_\infty$, as well as $\left\|\frac{1}{|\sigma|}\right\|_\infty < \infty$.
This leads to

$$\hat{V}_s = \hat{V}_T - \int_s^T \hat{Z}_r \,\mathrm{d}\tilde{W}_r - \int_s^T \left(\delta^{f,x}_r + \left(\delta^{f,z}_r - \tilde{\gamma}_r\right)\hat{Z}_r + \left(\delta^{f,y}_r - \delta^{\mu,x}_r\right)\hat{V}_r - \delta^{\mu,y}_r \hat{V}^2_r\right) \mathrm{d}r, \quad s \in [t,T],$$

where $\tilde{\gamma}$ is the $\mathbb{R}^{1\times(1\times d)}$ - valued version of γ.
Now, remember $\hat{V}_s = u_x(s,X_s)$ and the fact that X is a Gaussian process under $\tilde{\mathbb{P}}$ having the property of $X_s - X_r$ being independent of $\mathcal{F}_r$ (under $\tilde{\mathbb{P}}$) for $r \in [t,T]$, $s \in [r,T]$. Also,

$$\inf_{s,r \in [t,T]} \frac{\mathrm{Var}_{\tilde{\mathbb{P}}}(X_s - X_r)}{|s-r|} \geq \sigma_0^2 > 0,$$

if $\sigma_0 > 0$ is a lower bound of $|\sigma|$. Also,

$$\sup_{s \in [t,T]} \sup_{a,b \in \mathbb{R}} \left| \int_a^b u_x(s,x)\,\mathrm{d}x \right| \leq 2\|u\|_\infty \leq 2\|\xi\|_\infty + 2T\|f\|_\infty := C_u < \infty.$$

Furthermore, $\hat{Z}$ is in $BMO(\tilde{\mathbb{P}})$ according to Theorem A.1.6.

According to Lemma 3.2.5 the real-valued function φ defined as $\varphi_s := \|\hat{V}_s\|_\infty + \left(1 + \|\hat{V}_s\|_\infty^p\right)^{\frac{1}{p}} - 1$, $p = \frac{3}{2}$ satisfies

$$\varphi_s \leq \varphi_r \cdot \chi(\varphi_r \sqrt{r-s}) + \int_s^r C(1 + \varphi_q^\delta)\, \mathrm{d}q, \qquad s, r \in [t, T],\ s < r,$$

with constants $C > 0$, $\delta \in (1, 3)$ and a function χ which satisfies the properties from Lemma 3.2.4. C, δ, χ do not depend on $t \in I^b_{\max}$.

Although φ is defined on $[t, T]$ for a fixed $t \in (t^b_{\min}, T]$ we have $\|\hat{V}_s\|_\infty = \|u_x(s, X_s)\|_\infty = \|u_x(s, \cdot)\|_\infty$, for $s \in (t, T]$, since the probability distribution of X_s is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}$ for all $s \in (t, T]$. So, φ can also be defined as

$$\varphi_s = \|u_x(s, \cdot)\|_\infty + (1 + \|u_x(s, \cdot)\|_\infty^p)^{\frac{1}{p}},$$

for all $s \in (t^b_{\min}, T]$ and it will still satisfy

$$\varphi_s \leq \varphi_r \cdot \chi(\varphi_r \sqrt{r-s}) + \int_s^r C(1 + \varphi_q^\delta)\, \mathrm{d}q, \qquad s, r \in (t^b_{\min}, T],\ s < r,$$

which makes Lemma 3.2.4 applicable. According to (3.14) $\lim_{s \downarrow t^b_{\min}} \varphi_s = \infty$, which contradicts the statement of Lemma 3.2.4. $\square$

Chapter 4

Skorokhod Embedding via FBSDE

The Skorokhod embedding problem (SEP) stimulates research in probability theory now for over 50 years. Classically, the SEP is to find, for a given Brownian motion W and a probability measure ν, a stopping time τ such that W_τ has the law ν. It was first formulated and solved by Skorokhod [Sko61, Sko65] in 1961. Since then there appeared many different constructions for the stopping time τ and generalizations of the original problem in the literature. Just to name some of the most famous solutions to the SEP we refer to Root [Roo69], Rost [Ros71] and Azéma-Yor [AY79]. A comprehensive survey can be found in [Obł04].
Recently, the Skorokhod embedding steams additional interest because of its application in financial mathematics as for instance to obtain model-independent bounds on lookback options [Hob98] or on options on variance [CL10, CW13, OdR13]. An introduction to this close connection of the Skorokhod embedding problem and robust financial mathematics can be found in [Hob11].
In this chapter we construct a solution to the Skorokhod embedding problem for Gaussian process G of the form

$$G_t := G_0 + \int_0^t \alpha_s \,\mathrm{d}s + \int_0^t \beta_s \,\mathrm{d}W_s,$$

where $G_0 \in \mathbb{R}$ is some constant and $\alpha, \beta \colon [0,\infty) \to \mathbb{R}$ are deterministic and sufficiently smooth. Especially, this class of processes includes the Brownian motions with non-linear drift. The SEP for a Brownian motion with linear drift was first solved in the not so well-known technical report [Hal68] and 30 years later again in [GF00] and [Pes00]. The later techniques can also be extended to time-homogeneous diffusion as done in [PP01] and can be seen as generalization of the Azéma-Yor solution. However, to our best knowledge there exists no solution so far for the case of a Brownian motion with non-linear drift.
The spirit of our approach is related to an approach to the SEP by Bass [Bas83], who provided a solution to SEP for the Brownian motion. This approach was further developed for the Brownian motion with linear drift in [AHI08] and also for time-homogeneous diffusion in [AHS13]. Roughly speaking, it relies on the observation that the SEP may be viewed as the weak version of a stochastic control problem: We want to steer G in such a way that it takes the distribution of a prescribed law. This allows us to formulate the SEP in terms of a fully coupled Forward-Backward Stochastic Differential Equation (FBSDE).
In general, the dynamics of an FBSDE is classically given by

$$\begin{aligned} X_s &= X_0 + \int_0^s \mu(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_0^s \sigma(r, X_r, Y_r, Z_r)\,\mathrm{d}W_r, \\ Y_t &= \xi(X_T) - \int_t^T f(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_t^T Z_r\,\mathrm{d}W_r, \end{aligned}$$

where we assume for a moment that all objects are well-defined. In this chapter we focus on Markovian FBSDE meaning that all involved functions ξ, μ, σ, f are deterministic functions. Allowing that μ, σ, f are only locally Lipschitz in Z, we develop an existence, uniqueness and regularity theory. The results are very similar to the treatment of the Markovian case in Chapter 2, Section 2.5.1. However, since now we require Lipschitz continuity in Y the assumption of the boundedness of the terminal condition ξ can be dropped. We will refer to this situation as the MLLC case (standing for Modified Local Lipschitz Conditions) in order to distinguish it from the SLLC case of Section 2.5.1 of Chapter 2 (standing for Standard Local Lipschitz Conditions).
With these techniques at hand we solve the FBSDE corresponding to the SEP. In this way we first construct a weak solution to the Skorokhod embedding problem, i.e. we obtain a Gaussian process of the above form and an integrable random time such that our Gaussian process stopped at this time possesses the given distribution ν. Under suitable regularity on the given measure ν and the Gaussian process, this construction will be carried over to the given Gaussian process G thus solving the originally posed SEP.

The chapter is organized as follows: In Section 4.1 we relate the SEP to a coupled FBSDE. In Section 4.2 we sum up general results for decoupling fields of FBSDE and prove several statements for the Markovian case, which are not present in Chapter 2. The Skorokhod embedding problem is solved in Section 4.3, in its weak and in its strong version. The basis of our analysis is Lemma 4.3.1 which provides solvability of the coupled system. The main result is Theorem 4.3.6 together with Lemma 4.3.10, which in combination imply that our construction leads to strong solutions to the SEP if we assume that ν, α, β are sufficiently regular.

The key to proving Theorem 4.3.6 is Lemma 4.3.5, which connects the control process Z, which in turn defines the stopping time, to the decoupling field u. It also explains why regularity of u is fundamental to analyzing the properties of the (alleged) stopping time. In the subsequent analysis of continuity and differentiability properties of u we first show that u has bounded spatial derivatives up to order 3. Based on this we prove Lipschitz continuity in time of one of the spatial derivatives of u (see proof of Lemma 4.3.10), which is key to applying Theorem 4.3.6.
This analysis of spatial differentiability of u is based on the technique of aggregating the dynamics of u together with those of its (alleged) derivatives into one large multi-dimensional system and then showing its well-posedness using the theory from Section 4.2. In the second step one shows that some components of the joint decoupling field are indeed spatial derivatives of the other (see proofs of Theorems 4.3.7 and 4.3.9 for details).

4.1 FBSDE approach to the Skorokhod embedding problem

We consider a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$ large enough to carry a one-dimensional Brownian motion W. The filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ is assumed to be generated by the Brownian motion and augmented by $\mathbb{P}$-null sets. We also assume that $\mathcal{F} = \sigma\left(\bigcup_{t=0}^{\infty} \mathcal{F}_t\right)$.

Let us start by formulating the following version of the Skorokhod embedding problem **(SEP)**: For a given probability measure ν on $\mathbb{R}$ and a Gaussian process G on $[0,\infty)$ of the form

$$G_t := G_0 + \int_0^t \alpha_s \, \mathrm{d}s + \int_0^t \beta_s \, \mathrm{d}W_s, \tag{4.1}$$

where $G_0 \in \mathbb{R}$ is some predetermined constant and $\alpha, \beta \colon [0,\infty) \to \mathbb{R}$ are deterministic measurable processes such that $\int_0^t |\alpha_s| \, \mathrm{d}s + \int_0^t \beta_s^2 \, \mathrm{d}s < \infty$ for all $t \geq 0$, find

- an $(\mathcal{F}_t)$-stopping time τ s.t. $\mathbb{E}[\tau] < \infty$ together with

- a starting point $c \in \mathbb{R}$

such that $c + G_\tau$ has the law ν.

In order to have a truly stochastic problem, β should not vanish and ν should not be a Dirac measure. In fact we will assume that β is bounded away from zero later on.
Our method of solving this problem is based on the observation that the problem may be viewed as the weak version of a *stochastic control problem*: We want to steer G in such a way that it takes the distribution of a prescribed law. The spirit of our approach is related to an approach to the original Skorokhod embedding problem by Bass [Bas83] and later extended to the Brownian motion with linear drift in [AHI08]. The procedure of both papers can be briefly summarized and split in the following four steps.

1. Construct a function $g\colon \mathbb{R} \to \mathbb{R}$ such that $g(W_1)$ has the given law ν.

2. Use directly the martingale representation property of the Brownian motion for $\alpha \equiv 0$ and $\beta \equiv 1$ or BSDE techniques for $\alpha \equiv \kappa \neq 0$ and $\beta \equiv 1$ to solve

$$Y_t = g(W_1) - \kappa \int_t^1 Z_s^2 \,\mathrm{d}s - \int_t^1 Z_s \,\mathrm{d}W_s, \quad t \in [0,1]. \tag{4.2}$$

3. Apply for the above equation the random time-change of Dambis, Dubins and Schwarz in the quadratic variation scale $\int_0^\cdot Z_s^2 \,\mathrm{d}s$ to transform the martingale $\int_0^\cdot Z_s \,\mathrm{d}W_s$ into a Brownian motion B. This also provides us with a random time $\tilde{\tau} := \int_0^1 Z_s^2 \,\mathrm{d}s$ fulfilling $B_{\tilde{\tau}} + \kappa\tilde{\tau} + Y_0 = g(W_1)$, which is why $B_{\tilde{\tau}} + \kappa\tilde{\tau} + Y_0$ has the law ν.

4. Show that $\tilde{\tau}$ is a stopping time with respect to the filtration generated by B through an explicit characterization using the unique solution of an ordinary differential equation. With this description transform the embedding with respect to B into one with respect to the original Brownian motion W to obtain the stopping time τ as the analogue to $\tilde{\tau}$.

The **first step** of the illustrated algorithm is fairly easy. Let $F\colon \mathbb{R} \to [0,1]$ such that $F(x) := \nu((-\infty, x])$ be the cumulative distribution function associated with ν and define $F^{-1} : (0,1) \to \mathbb{R}$ via

$$F^{-1}(y) := \inf\{x \in \mathbb{R} \mid F(x) \geq y\}.$$

Denoting by Φ the distribution function of the standard normal distribution, we define $g\colon \mathbb{R} \to \mathbb{R}$ as $g(x) := F^{-1}(\Phi(x))$. It is straightforward to proof that g has the following properties:

Lemma 4.1.1. *The function g is measurable and non-decreasing. Moreover $g(W_1)$ has the law ν. Also, if ν is not a Dirac measure, then g is not identically constant.*

Proof. Since Φ and F^{-1} are measurable and non-decreasing, their composition g is also measurable and non-decreasing. In order to see that $g(W_1)$ has the law ν, note that

$$\mathbb{P}(g(W_1) \leq x) = \mathbb{P}(F^{-1}(\Phi(W_1)) \leq x) = \mathbb{P}(W_1 \leq \Phi^{-1}(F(x))) = \Phi(\Phi^{-1}(F(x))) = F(x)$$

for all $x \in \mathbb{R}$.
Clearly, g can only be constant if F^{-1} is constant, which can only happen if F assumes values in $\{0,1\}$, which happens only in case ν is a Dirac measure. □

Now, define a measurable function $\hat{\delta}\colon [0,\infty) \to \mathbb{R}$ via

$$\hat{\delta}(t) := G_0 + \int_0^t \alpha_s \,\mathrm{d}s$$

such that $G_t = \hat{\delta}(t) + \int_0^t \beta_s \,\mathrm{d}W_s$. Obviously $\hat{\delta}$ is weakly differentiable. Conversely, for every weakly differentiable function $\hat{\delta}\colon [0,\infty) \to \mathbb{R}$ we can set $G_0 := \hat{\delta}(0)$ and $\alpha_s := \hat{\delta}'(s)$.

Furthermore, define $H\colon [0,\infty) \to [0,\infty)$ via

$$H(t) := \int_0^t \beta_s^2 \,\mathrm{d}s.$$

Note that H is weakly differentiable, monotonically increasing and starts at 0. If we assume that β is bounded away from 0, H becomes strictly increasing and invertible such that the inverse function H^{-1} is monotonically increasing and Lipschitz continuous. In this case we can define

$$\delta := \hat{\delta} \circ H^{-1}.$$

If $\beta \equiv 1$, then $H = \mathrm{Id}$ and, thus, $\delta = \hat{\delta}$.

For the **second step** we assume that β is bounded away from 0 and observe that the random time change, which turns the martingale $\int_0^\cdot Z_s \,\mathrm{d}W_s$ into a Gaussian process of the form $\int_0^\cdot \beta_s \,\mathrm{d}B_s$ simultaneously turns the scale process $\int_0^\cdot Z_s^2 \,\mathrm{d}s$ into $\int_0^\cdot \beta_s^2 \,\mathrm{d}s = H$. This means that we have to modify the classical martingale representation of $g(W_1)$ to

$$g(W_1) - \hat{\delta}\left(H^{-1}\left(\int_0^1 Z_s^2 \,\mathrm{d}s\right)\right) - \mathbb{E}\left[g(W_1) - \hat{\delta}\left(H^{-1}\left(\int_0^1 Z_s^2 \,\mathrm{d}s\right)\right)\right] = \int_0^1 Z_s \,\mathrm{d}W_s,$$

which amounts to finding a solution (Y, Z) to the equation

$$Y_t = g(W_1) - \delta\left(\int_0^1 Z_s^2 \,\mathrm{d}s\right) - \int_t^1 Z_s \,\mathrm{d}W_s, \quad t \in [0,1]. \tag{4.3}$$

For $\delta(t) \equiv 0$ this would be just the usual martingale representation with respect to the Brownian motion. Also, for a linear drift $\delta(t) = \kappa t$ and $\beta \equiv 1$ equation (4.3) can be rewritten as

$$\tilde{Y}_t := Y_t + \kappa \int_0^t Z_s^2 \,\mathrm{d}s = g(W_1) - \kappa \int_t^1 Z_s^2 \,\mathrm{d}s - \int_t^1 Z_s \,\mathrm{d}W_s, \quad t \in [0,1],$$

which is exactly the BSDE (4.2) related to the SEP as stated in [AHI08]. In the case of a Brownian motion with general drift, equation (4.3) would be a BSDE with time-delayed terminal condition. Unfortunately, the theory of BSDE with time-delay as introduced by Delong and Imkeller in [DI10] and extended by Delong in [Del12] for time-delayed terminal conditions approaches in our situation its limits. Instead, we will understand equation (4.3) as an FBSDE and develop new techniques to solve it. This will be done in Sections 4.2 and 4.3. Before we tackle the solvability of equation (4.3), we show that it really leads to the desired result in the **third step** of our algorithm. To be mathematically rigorous we introduce

- $\mathbb{S}^2(\mathbb{R})$ as the space of all progressively measurable processes $Y\colon \Omega \times [0,1] \to \mathbb{R}$ satisfying $\sup_{t\in[0,1]} \mathbb{E}[|Y_t|^2] < \infty$,
- $\mathbb{H}^2(\mathbb{R})$ as the space of all progressively measurable processes $Z\colon \Omega \times [0,1] \to \mathbb{R}$ satisfying $\mathbb{E}[\int_0^1 |Z_t|^2 \,\mathrm{d}t] < \infty$,

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}$.

For the rest of the chapter we assume that β is **bounded away from** 0, i.e. $\inf_{s\in[0,\infty)}|\beta_s| > 0$.

Lemma 4.1.2. *Suppose $(Y,Z)\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})$ is a solution of (4.3). Then there exist a Brownian motion B and a random time $\tilde{\tau}$ with $\mathbb{E}[\tilde{\tau}]<\infty$ such that*

$$Y_0 + G_0 + \int_0^{\tilde{\tau}} \alpha_s \,\mathrm{d}s + \int_0^{\tilde{\tau}} \beta_s \,\mathrm{d}B_s = g(W_1).$$

Proof. Note that (Y_t) is a martingale with quadratic variation process $\int_0^t Z_s^2\,\mathrm{d}s$ for $t\in[0,1]$, since $Z\in\mathbb{H}^2(\mathbb{R})$. Now, choose another Brownian motion $\tilde{B}$ which is independent of Y. If necessary, we extend our probability space such that it accommodates the Brownian motion $\tilde{B}$. Set $\tilde{\tau} := H^{-1}\left(\int_0^1 Z_s^2\,\mathrm{d}s\right)$ and define the time-change of Dambis, Dubins and Schwarz type by

$$\sigma_r := \begin{cases} \inf\left\{t\geq 0 \,\middle|\, \int_0^t Z_s^2\,\mathrm{d}s > \int_0^r \beta_s^2\,\mathrm{d}s\right\} & \text{if } 0\leq r<\tilde{\tau} \\ 1 & \text{if } r\geq\tilde{\tau}. \end{cases}$$

Observe that the condition $r<\tilde{\tau}$ is equivalent to $\int_0^r \beta_s^2\,\mathrm{d}s < \int_0^1 Z_s^2\,\mathrm{d}s$. Since Y_{σ_r} is a continuous martingale with quadratic variation $H(r)=\int_0^r\beta_s^2\,\mathrm{d}s$, we can define a Brownian motion B by

$$B_r := \tilde{B}_r - \tilde{B}_{r\wedge\tilde{\tau}} + \int_0^{r\wedge\tilde{\tau}} \frac{1}{\beta_s}\,\mathrm{d}Y_{\sigma_s}, \quad 0\leq r<\infty.$$

We find

$$\int_0^{\tilde{\tau}} \beta_s\,\mathrm{d}B_s + \hat{\delta}(\tilde{\tau}) + Y_0 = Y_1 - Y_0 + \delta\left(\int_0^1 Z_s^2\,\mathrm{d}s\right) + Y_0 = g(W_1)$$

and further

$$\mathbb{E}[\tilde{\tau}] = \mathbb{E}\left[H^{-1}\left(\int_0^1 Z_s^2\,\mathrm{d}s\right)\right] < \infty,$$

where we used that $Z\in\mathbb{H}^2(\mathbb{R})$ and that H^{-1} is Lipschitz continuous. □

An immediate consequence of this lemma is: If we have a solution $(Y,Z)\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})$ of equation (4.3), we obtain a *weak solution* to the Skorokhod embedding problem, i.e. we constructed a Gaussian process of the form (4.1), a starting point c, and an integrable random time such that our Gaussian process stopped at this time possesses a given distribution.
At a first glance equation (4.3) might look easy but it turns out that we, in fact, deal with a fully coupled FBSDE with a non-Lipschitz continuous coefficient in the forward equation.

4.2 Decoupling fields for fully coupled FBSDEs

Let us in the following recall the key results of the theory of decoupling fields. Although later on, in Section 4.2.2, we will focus on the Markovian case, which means that all involved coefficients of the FBSDE will be purely deterministic, let us recall the general theory first. We will give a summary of the main results of Section 2.5.

4.2.1 General decoupling fields

For a fixed time horizon $T > 0$, we consider a complete filtered probability space

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}),$$

where $\mathcal{F}_0 = \sigma(p) \vee \mathcal{N}$ for some $p : \Omega \to S$ and a polish space S, $\mathcal{N}$ are the null sets, , $(W_t)_{t\in[0,T]}$ is a d-dimensional Brownian motion independent of $\mathcal{F}_0$, and $\mathcal{F}_t = \sigma(\mathcal{F}_0, (W_s)_{s\in[0,t]})$ with $\mathcal{F} = \mathcal{F}_T$. The dynamics of an FBSDE is classically given by

$$\begin{aligned} X_s &= X_0 + \int_0^s \mu(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_0^s \sigma(r, X_r, Y_r, Z_r)\,\mathrm{d}W_r, \\ Y_t &= \xi(X_T) - \int_t^T f(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_t^T Z_r\,\mathrm{d}W_r, \end{aligned}$$

for $s, t \in [0, T]$ and $X_0 \in \mathbb{R}^n$, where $(\xi, (\mu, \sigma, f))$ are measurable functions. More precisely,

$$\begin{aligned} &\xi\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^m, && \mu\colon [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^n, \\ &\sigma\colon [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^{n\times d}, && f\colon [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^m, \end{aligned}$$

for $n, m, d \in \mathbb{N}$. Throughout the whole chapter μ, σ and f are assumed to be progressively measurable with respect to $(\mathcal{F}_t)_{t\in[0,T]}$, i.e. $\mu\mathbf{1}_{[0,t]}, \sigma\mathbf{1}_{[0,t]}, f\mathbf{1}_{[0,t]}$ are $\mathcal{B}([0,T]) \otimes \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^{m\times d})$-measurable for all $t \in [0, T]$.
A decoupling field comes with an even richer structure than just a classical solution.

Definition 4.2.1. Let $t \in [0, T]$. A function $u\colon [t, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ with $u(T, \omega, \cdot) = \xi(\omega, \cdot)$ for a.a. $\omega \in \Omega$ is called a *decoupling field* for $(\xi, (\mu, \sigma, f))$ on $[t, T]$ if for all $t_1, t_2 \in [t, T]$ with $t_1 \le t_2$ and any $\mathcal{F}_{t_1}$-measurable $X_{t_1}\colon \Omega \to \mathbb{R}^n$ there exist progressive processes (X, Y, Z) on $[t_1, t_2]$ such that

$$\begin{aligned} X_s &= X_{t_1} + \int_{t_1}^s \mu(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_{t_1}^s \sigma(r, X_r, Y_r, Z_r)\,\mathrm{d}W_r, \\ Y_s &= Y_{t_2} - \int_s^{t_2} f(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^{t_2} Z_r\,\mathrm{d}W_r, \\ Y_s &= u(s, X_s), \end{aligned} \tag{4.4}$$

a.s. for all $s \in [t_1, t_2]$. In particular, we want all integrals to be well-defined and X, Y, Z to have values in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$, respectively.

Some remarks about this definition:

- The first equation in (4.4) is called the *forward equation*, the second the *backward equation* and the third will be referred to as the *decoupling condition.*
- The above requirement that X should start at X_{t_1} is referred to as the *initial condition.* X_{t_1} is also sometimes referred to as the *initial value.*
- Note that if $t_2 = T$, we get $Y_T = \xi(X_T)$ a.s. as a consequence of the decoupling condition together with $u(T, \omega, \cdot) = \xi(\omega, \cdot)$ for a.a. $\omega \in \Omega$.
- If $t_2 = T$, we can say that a triple (X, Y, Z) solves the FBSDE, meaning that it satisfies the forward and the backward equation, together with $Y_T = \xi(X_T)$. This relationship $Y_T = \xi(X_T)$ is referred to as the *terminal condition.*

In contrast to classical solutions of FBSDE, decoupling fields on different intervals can be pasted together.

Lemma 4.2.2 (Lemma 2.1.2 in Chapter 2)**.** *Let u be a decoupling field for $(\xi,(\mu,\sigma,f))$ on $[t,T]$ and $\tilde{u}$ be a decoupling field for $(u(t,\cdot),(\mu,\sigma,f))$ on $[s,t]$, for $0\leq s<t<T$. Then, the map $\hat{u}$ given by $\hat{u}:=\tilde{u}\mathbf{1}_{[s,t]}+u\mathbf{1}_{(t,T]}$ is a decoupling field for $(\xi,(\mu,\sigma,f))$ on $[s,T]$.*

We want to remark that, if u is a decoupling field and $\tilde{u}$ is a modification of u, i.e. for each $s\in[t,T]$ the functions $u(s,\omega,\cdot)$ and $\tilde{u}(s,\omega,\cdot)$ coincide for almost all $\omega\in\Omega$, then $\tilde{u}$ is also a decoupling field to the same problem. So, u could also be referred to as a class of modifications. Some of the representatives of the class might be progressively measurable, others not. As we see below a progressively measurable representative does exist if the decoupling field is Lipschitz continuous in x:

Lemma 4.2.3 (Lemma 2.1.3 in Chapter 2)**.** *Let $u\colon[t,T]\times\Omega\times\mathbb{R}^n\to\mathbb{R}^m$ be a decoupling field to $(\xi,(\mu,\sigma,f))$ which is Lipschitz continuous in $x\in\mathbb{R}^n$ in the weak sense that there exists a constant $L>0$ s.t. for every $s\in[t,T]$:*

$$|u(s,\omega,x)-u(s,\omega,x')|\leq L|x-x'|\qquad\forall x,x'\in\mathbb{R}^n,\quad\text{for a.a. }\omega\in\Omega.$$

Then u has a modification $\tilde{u}$ which is progressively measurable.

Let $I\subseteq[0,T]$ be an interval and $u:I\times\Omega\times\mathbb{R}^n\to\mathbb{R}^m$ a map such that $u(s,\cdot)$ is measurable for every $s\in I$. We define

$$L_{u,x}:=\sup_{s\in I}\inf\{L\geq 0\,|\,\text{for a.a. }\omega\in\Omega:|u(s,\omega,x)-u(s,\omega,x')|\leq L|x-x'|\text{ for all }x,x'\in\mathbb{R}^n\}.$$

where $\inf\emptyset:=\infty$. We also set $L_{u,x}:=\infty$ if $u(s,\cdot)$ is not measurable for every $s\in I$. One can show that $L_{u,x}<\infty$ is equivalent to u having a modification, which is truly Lipschitz continuous in $x\in\mathbb{R}^n$.

We denote by $L_{\sigma,z}$ the Lipschitz constant of σ w.r.t. the dependence on the last component z (and w.r.t. the Frobenius norms on $\mathbb{R}^{m\times d}$ and $\mathbb{R}^{n\times d}$). We set $L_{\sigma,z}=\infty$ if σ is not Lipschitz continuous in z.
By $L_{\sigma,z}^{-1}=\frac{1}{L_{\sigma,z}}$ we mean $\frac{1}{L_{\sigma,z}}$ if $L_{\sigma,z}>0$ and ∞ otherwise.

Definition 4.2.4. Let $u\colon[t,T]\times\Omega\times\mathbb{R}^n\to\mathbb{R}^m$ be a decoupling field to $(\xi,(\mu,\sigma,f))$. We say that u is *weakly regular* if $L_{u,x}<L_{\sigma,z}^{-1}$ and $\sup_{s\in[t,T]}\|u(s,\cdot,0)\|_\infty<\infty$.

This is a natural definition due to Lemma 4.2.3, in practice, however, it is important to have explicit knowledge about the regularity of (X,Y,Z). For instance, it is important to know in which spaces the processes live, and how they react to changes in the initial value. Specifically, it can be very useful to have differentiability of (X,Y,Z) w.r.t. the initial value.
In the following we need further notation. For an integrable real valued random variable F the expression $\mathbb{E}_t[F]$ refers to $\mathbb{E}[F|\mathcal{F}_t]$, while $\mathbb{E}_{\hat{t},\infty}[F]$ refers to $\operatorname{ess\,sup}\mathbb{E}[F|\mathcal{F}_t]$, which might be ∞, but is always well-defined as the infimum of all constants $c\in[-\infty,\infty]$ such that $\mathbb{E}[F|\mathcal{F}_t]\leq c$ a.s.
As usual, we write $\|F\|_\infty$ for the essential supremum of $|F|$, for an arbitrary measurable F.

Definition 4.2.5. Let $u\colon[t,T]\times\Omega\times\mathbb{R}^n\to\mathbb{R}^m$ be a weakly regular decoupling field to $(\xi,(\mu,\sigma,f))$. We call u *strongly regular* if for all fixed $t_1,t_2\in[t,T]$, $t_1\leq t_2$, the processes (X,Y,Z) arising in (4.4) are a.e unique and satisfy

$$\sup_{s\in[t_1,t_2]}\mathbb{E}_{t_1,\infty}[|X_s|^2]+\sup_{s\in[t_1,t_2]}\mathbb{E}_{t_1,\infty}[|Y_s|^2]+\mathbb{E}_{t_1,\infty}\left[\int_{t_1}^{t_2}|Z_s|^2\,\mathrm{d}s\right]<\infty,\tag{4.5}$$

for each constant initial value $X_{t_1} = x \in \mathbb{R}^n$. In addition they must be measurable as functions of (x, s, ω) and even weakly differentiable w.r.t. $x \in \mathbb{R}^n$ such that for every $s \in [t_1, t_2]$ the mappings X_s and Y_s are measurable functions of (x, ω) and even weakly differentiable w.r.t. x such that

$$
\begin{aligned}
&\operatorname{ess\,sup}_{x\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \sup_{s\in[t_1,t_2]} \mathbb{E}_{t_1,\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}x}X_s\right|_v^2\right] < \infty,\\
&\operatorname{ess\,sup}_{x\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \sup_{s\in[t_1,t_2]} \mathbb{E}_{t_1,\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}x}Y_s\right|_v^2\right] < \infty,\\
&\operatorname{ess\,sup}_{x\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \mathbb{E}_{t_1,\infty}\left[\int_{t_1}^{t_2}\left|\frac{\mathrm{d}}{\mathrm{d}x}Z_s\right|_v^2 \mathrm{d}s\right] < \infty.
\end{aligned}
\tag{4.6}
$$

We say that a decoupling field u on $[t, T]$ is *strongly regular* on a subinterval $[t_1, t_2] \subseteq [t, T]$ if u restricted to $[t_1, t_2]$ is a strongly regular decoupling field for $(u(t_2, \cdot), (\mu, \sigma, f))$.

Strong regularity is a fundamental concept in our theory. It allows to work with weak derivatives and apply the rules of Lemmas A.2.4 to A.2.8 in particular. Consult Section 2.1.2 of Chapter 2 for more on the subject of weak derivatives.

Under certain conditions a rich existence, uniqueness and regularity theory for decoupling fields can be developed. We will summarize the main results, which are proven in Chapter 2:

Assumption (SLC): $(\xi, (\mu, \sigma, f))$ satisfies *standard Lipschitz conditions* SLC if

1. (μ, σ, f) are Lipschitz continuous in (x, y, z) with some Lipschitz constant L,
2. $\|(|\mu| + |f| + |\sigma|)(\cdot, \cdot, 0, 0, 0)\|_\infty < \infty$,
3. $\xi\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ is measurable such that $\|\xi(\cdot, 0)\|_\infty < \infty$ and $L_{\xi,x} < L_{\sigma,z}^{-1}$.

Theorem 4.2.6 (Theorem 2.2.1 in Chapter 2)**.** *Suppose $(\xi, (\mu, \sigma, f))$ satisfies SLC. Then there exists a time $t \in [0, T)$ such that $(\xi, (\mu, \sigma, f))$ has a unique (up to modification) decoupling field u on $[t, T]$ with $L_{u,x} < L_{\sigma,z}^{-1}$ and $\sup_{s\in[t,T]} \|u(s, \cdot, 0)\|_\infty < \infty$.*

A brief discussion of existence and uniqueness of classical solutions for small intervals can be found in Remark 2.2.4 in Chapter 2. For later reference we give the following remarks (cf. Remark 2.2.2 and 2.2.3 in Chapter 2).

Remark 4.2.7. It can be observed from the proof of the above theorem that the supremum of all $h = T - t$, with t satisfying the properties required in Theorem 4.2.6 can be bounded away from 0 by a bound, which depends only on

- the Lipschitz constant L of μ, σ and f w.r.t. the last 3 components, T
- L_ξ and $L_\xi \cdot L_{\sigma,z} < 1$,

and which is monotonically decreasing in these values.

Remark 4.2.8. It can also be observed from the proof that our decoupling field u on $[t, T]$ satisfies

$$L_{u(s,\cdot),x} \le L_{\xi,x} + C(T - s)^{\frac{1}{4}},$$

where C is some constant which does not depend on $s \in [t, T]$. More precisely, C depends only on T, L, $L_{\xi,x}$, $L_{\xi,x}L_{\sigma,z}$ and is monotonically increasing in these values.

We can systematically extend this local theory to obtain global results. This is based on a simple argument which we will refer to as *small interval induction*.

Lemma 4.2.9 (Lemma 2.5.1 and 2.5.2 in Chapter 2). *Let $T_1 < T_2$ be real numbers and let $S \subseteq [T_1, T_2]$.*

1. Forward: *If $T_1 \in S$ and there exists an $h > 0$ s.t. $[s, s+h] \cap [T_1, T_2] \subseteq S$ for all $s \in S$, then $S = [T_1, T_2]$ and in particular $T_2 \in S$.*

2. Backward: *If $T_2 \in S$ and there exists an $h > 0$ s.t. $[s-h, s] \cap [T_1, T_2] \subseteq S$ for all $s \in S$, then $S = [T_1, T_2]$ and in particular $T_1 \in S$.*

Using these simple results we obtain *global uniqueness* and *global regularity* of a decoupling field.

Theorem 4.2.10 (Corollary 2.5.3 and 2.5.4 in Chapter 2). *Suppose that $(\xi, (\mu, \sigma, f))$ satisfies SLC.*

1. Global uniqueness: *If there are two weakly regular decoupling fields $u^{(1)}, u^{(2)}$ to the corresponding problem on some interval $[t, T]$, then we have $u^{(1)} = u^{(2)}$ up to modifications.*

2. Global regularity: *If there exists a weakly regular decoupling field u to this problem on some interval $[t, T]$, then u is strongly regular.*

Notice that Theorem 4.2.10 only provides uniqueness of weakly regular decoupling fields, not uniqueness of processes (X, Y, Z) solving the FBSDE in the classical sense. However, using global regularity in Theorem 4.2.10 one can show:

Corollary 4.2.11 (Corollary 2.5.5 in Chapter 2). *Let $(\xi, (\mu, \sigma, f))$ fulfill SLC. If there exists a weakly regular decoupling field u on some interval $[t, T]$, then for any initial condition $X_t = x \in \mathbb{R}^n$ there is a unique solution (X, Y, Z) of the FBSDE on $[t, T]$ satisfying*

$$\sup_{s\in[t,T]} \mathbb{E}_{0,\infty}[|X_s|^2] + \sup_{s\in[t,T]} \mathbb{E}_{0,\infty}[|Y_s|^2] + \mathbb{E}_{0,\infty}\left[\int_t^T |Z_s|^2 \,\mathrm{d}s\right] < \infty.$$

4.2.2 Markovian decoupling fields

A problem given by $(\xi, (\mu, \sigma, f))$ is said to be *Markovian* if these four functions are deterministic, that is, they depend only on (t, x, y, z). In the Markovian situation we can somewhat relax the Lipschitz continuity assumption and still obtain local existence together with uniqueness. What makes the Markovian case so special is the property

$$"Z_s = u_x(s, X_s) \cdot \sigma(s, X_s, Y_s, Z_s)",$$

which comes from the fact that u will also be deterministic. This property allows us to bound Z by a constant if we assume that σ is bounded.

Lemma 4.2.12 (Lemma 2.5.13 in Chapter 2). *Let μ, σ, f, ξ satisfy SLC and assume in addition that they are deterministic. Assume that we have a weakly regular decoupling field u on an interval $[t, T]$. Then u is deterministic in the sense that it has a modification which is a function of $(r, x) \in [t, T] \times \mathbb{R}^n$ only.*

An application of Lemma 4.2.12 is the following quite fundamental result.

Lemma 4.2.13 (Lemma 2.5.14 in Chapter 2). *Let $(\xi, (\mu, \sigma, f))$ satisfy SLC and suppose that these functions are deterministic. Let u be a weakly regular decoupling field on an interval $[t, T]$. Choose $t_1 < t_2$ from $[t, T]$ and an initial condition X_{t_1}. Then the corresponding Z satisfies $\|Z\|_\infty \le L_{u,x} \cdot \|\sigma\|_\infty$. If $\|Z\|_\infty < \infty$, we also have $\|Z\|_\infty \le L_{u,x} \|\sigma(\cdot, \cdot, \cdot, 0)\|_\infty (1 - L_{u,x} L_{\sigma,z})^{-1}$.*

Next we investigate the continuity of u as a function of time and space.

Lemma 4.2.14 (Lemma 2.5.15 in Chapter 2)**.** *Assume that μ, σ, f have linear growth in (x, y) in the sense*

$$(|\mu| + |\sigma| + |f|)\,(t, \omega, x, y, z) \leq C\,(1 + |x| + |y|) \quad \forall (t, x, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d},$$

for a.a. $\omega \in \Omega$, where $C \in [0, \infty)$ is some constant.
If u is a strongly regular and deterministic decoupling field to $(\xi, (\mu, \sigma, f))$ on an interval $[t, T]$, then u is continuous in the sense that it has a modification which is a continuous function on $[t, T] \times \mathbb{R}^n$.

This boundedness of Z in the Markovian case motivates the following definition, which will allow us to develop a theory for non-Lipschitz problems:

Definition 4.2.15. Let $t \in [0, T]$. We call a function $u\colon [t, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ with $u(T, \omega, \cdot) = \xi(\omega, \cdot)$ for a.a. $\omega \in \Omega$ a *Markovian decoupling field* for $(\xi, (\mu, \sigma, f))$ on $[t, T]$ if for all $t_1, t_2 \in [t, T]$ with $t_1 \leq t_2$ and any $\mathcal{F}_{t_1}$-measurable $X_{t_1}\colon \Omega \to \mathbb{R}^n$ there exist progressive processes (X, Y, Z) on $[t_1, t_2]$ such that the equations in (4.4) hold a.s. for all $s \in [t_1, t_2]$ and additionally $\|Z\|_\infty < \infty$.

We want to remark that a Markovian decoupling field is always a decoupling field in the standard sense as well. The only difference is that we are only interested in triples (X, Y, Z), where Z is bounded up to a null set.
Regularity for Markovian decoupling fields is defined very similarly to standard regularity:

Definition 4.2.16. Let $u\colon [t, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ be a Markovian decoupling field to $(\xi, (\mu, \sigma, f))$.

- We call u *weakly regular*, if $L_{u,x} < L_{\sigma,z}^{-1}$ and $\sup_{s\in[t,T]} \|u(s, \cdot, 0)\|_\infty < \infty$.
- We call a weakly regular u *strongly regular* if for all fixed $t_1, t_2 \in [t, T]$, $t_1 \leq t_2$, the processes (X, Y, Z) arising in the defining property of a Markovian decoupling field are a.e. unique for each *constant* initial value $X_{t_1} = x \in \mathbb{R}^n$ and satisfy (4.5). In addition they must be measurable as functions of (x, s, ω) and even weakly differentiable w.r.t. $x \in \mathbb{R}^n$ such that for every $s \in [t_1, t_2]$ the mappings X_s and Y_s are measurable functions of (x, ω) and even weakly differentiable w.r.t. x such that (4.6) holds.
- We say that a Markovian decoupling field u on $[t, T]$ is strongly regular on a subinterval $[t_1, t_2] \subseteq [t, T]$ if u restricted to $[t_1, t_2]$ is a strongly regular Markovian decoupling field for $(u(t_2, \cdot), (\mu, \sigma, f))$.

Now, we define a class of problems for which an existence and uniqueness theory will be developed:

Assumption (MLLC): $(\xi, (\mu, \sigma, f))$ fulfills *modified local Lipschitz conditions* MLLC if

1. the functions (μ, σ, f) are
   (a) deterministic,
   (b) Lipschitz continuous in x, y, z on sets of the form $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times B$, where $B \subset \mathbb{R}^{m\times d}$ is an arbitrary bounded set,
   (c) and fulfill $\|\mu(\cdot, 0, 0, 0)\|_\infty, \|f(\cdot, 0, 0, 0)\|_\infty, \|\sigma(\cdot, \cdot, \cdot, 0)\|_\infty, L_{\sigma,z} < \infty$,
2. $\xi\colon \mathbb{R}^n \to \mathbb{R}^m$ satisfies $L_{\xi,x} < L_{\sigma,z}^{-1}$.

We start by providing a local existence result.

Theorem 4.2.17. *Let $(\xi,(\mu,\sigma,f))$ satisfy MLLC. Then there exists a time $t\in[0,T)$ such that $(\xi,(\mu,\sigma,f))$ has a unique weakly regular Markovian decoupling field u on $[t,T]$. This u is also strongly regular, deterministic, continuous and satisfies $\sup_{t_1,t_2,X_{t_1}}\|Z\|_\infty<\infty$, where $t_1<t_2$ are from $[t,T]$ and X_{t_1} is an initial value (see the definition of a Markovian decoupling field for the meaning of these variables).*

Proof. For any constant $H>0$ let $\chi_H\colon\mathbb{R}^{m\times d}\to\mathbb{R}^{m\times d}$ be defined as

$$\chi_H(z):=\mathbf{1}_{\{|z|<H\}}z+\frac{H}{|z|}\mathbf{1}_{\{|z|\geq H\}}z.$$

It is easy to check that χ_H is Lipschitz continuous with Lipschitz constant $L_{\chi_H}=1$ and bounded by H. Furthermore, we have $\chi_H(z)=z$ if $|z|\leq H$. We implement an "inner cutoff" by defining (μ_H,σ_H,f_H) via $\mu_H(t,x,y,z):=\mu(t,x,y,\chi_H(z))$, etc.
The boundedness of χ_H together with its Lipschitz continuity makes μ_H,σ_H,f_H Lipschitz continuous with some Lipschitz constant L_H. Furthermore, $L_{\sigma_H,z}\leq L_{\sigma,z}$. Also, μ_H,σ_H,f_H have linear growth in y,z as required by Lemma 4.2.14. According to Theorem 4.2.6 we know that the problem given by $(\xi,(\mu_H,\sigma_H,f_H))$ has a unique weakly regular decoupling field u on some small interval $[t',T]$ where $t'\in[0,T)$. We also know that this u is strongly regular (Theorem 4.2.10), deterministic (by Lemma 4.2.12) and continuous (by Lemma 4.2.14).
We will show that for sufficiently large H and $t\in[t',T)$ it will also be a Markovian decoupling field to the problem $(\xi,(\mu,\sigma,f))$. Using Remark 4.2.8

$$L_{u(t,\cdot),x}\leq L_{\xi,x}+C_H(T-t)^{\frac{1}{4}}\quad\forall t\in[t',T],$$

where $C_H<\infty$ is some constant, which does not depend on $t\in[t',T]$. For any $t_1\in[t',T]$ and $\mathcal{F}_{t_1}$ - measurable initial value X_{t_1} consider the corresponding unique X,Y,Z on $[t_1,T]$ satisfying the forward equation, the backward equation and the decoupling condition for μ_H,σ_H,f_H and u. Using Lemma 4.2.13 we have $\|Z\|_\infty\leq L_{u,x}\|\sigma_H\|_\infty\leq L_{u,x}\left(\|\sigma(\cdot,\cdot,\cdot,0)\|_\infty+L_{\sigma,z}H\right)<\infty$ and therefore

$$\begin{aligned}\|Z\|_\infty\leq\frac{\sup_{s\in[t_1,T]}L_{u(s,\cdot),x}\cdot\|\sigma(\cdot,\cdot,\cdot,0)\|_\infty}{1-\sup_{s\in[t_1,T]}L_{u(s,\cdot),x}L_{\sigma,z}}&\leq\frac{\left(L_{\xi,x}+C_H(T-t_1)^{\frac{1}{4}}\right)\cdot\|\sigma(\cdot,\cdot,\cdot,0)\|_\infty}{1-L_{\xi,x}L_{\sigma,z}-L_{\sigma,z}C_H(T-t_1)^{\frac{1}{4}}}=\\&=\frac{L_{\xi,x}\|\sigma(\cdot,\cdot,\cdot,0)\|_\infty}{1-L_{\xi,x}L_{\sigma,z}-L_{\sigma,z}C_H(T-t_1)^{\frac{1}{4}}}+\frac{C_H(T-t_1)^{\frac{1}{4}}\cdot\|\sigma(\cdot,\cdot,\cdot,0)\|_\infty}{1-L_{\xi,x}L_{\sigma,z}-L_{\sigma,z}C_H(T-t_1)^{\frac{1}{4}}}\end{aligned}\tag{4.7}$$

for $T-t_1$ small enough.
Now, we only need to

- choose H large enough such that $\frac{L_{\xi,x}\|\sigma(\cdot,\cdot,\cdot,0)\|_\infty}{1-L_{\xi,x}L_{\sigma,z}}$ becomes smaller $\frac{H}{4}$
- and then in the second step choose t close enough to T, such that
  - $L_{\sigma,z}C_H(T-t)^{\frac{1}{4}}$ becomes smaller $\frac{1}{2}\left(1-L_{\xi,x}L_{\sigma,z}\right)$,
  - $\frac{C_H\|\sigma(\cdot,\cdot,\cdot,0)\|_\infty(T-t)^{\frac{1}{4}}}{1-L_{\xi,x}L_{\sigma,z}}$ becomes smaller than $\frac{H}{4}$.

Considering (4.7) this implies that if $t_1\in[t,T]$ the process Z a.e. does not leave the region in which the cutoff is "passive", i.e. the ball of radius H. Therefore, u restricted to the interval $[t,T]$ is a decoupling field to $(\xi,(\mu,\sigma,f))$, not just to $(\xi,(\mu_H,\sigma_H,f_H))$. It is even a Markovian decoupling field due to the boundedness of our Z. As a Markovian decoupling field it is weakly regular, because it is weakly regular as a decoupling field to $(\xi,(\mu_H,\sigma_H,f_H))$.

Uniqueness: Assume there is another weakly regular Markovian decoupling field $\tilde{u}$ to $(\xi, (\mu, \sigma, f))$ on $[t, T]$. Choose a $t_1 \in [t, T]$ and an $x \in \mathbb{R}^n$ as initial condition $X_{t_1} = x$ and consider the corresponding processes $\tilde{X}, \tilde{Y}, \tilde{Z}$, which satisfy the corresponding FBSDE on $[t_1, T]$, together with the decoupling condition via $\tilde{u}$. At the same time consider X, Y, Z solving the same FBSDE on $[t_1, T]$, but associated with the Markovian decopling field u. Since $\tilde{Z}, Z$ are bounded, the two triples $(\tilde{X}, \tilde{Y}, \tilde{Z})$ and (X, Y, Z) also solve the Lipschitz FBSDE given by $(\xi, (\mu_H, \sigma_H, f_H))$ on $[t_1, T]$ for H large enough. The two conditions $\tilde{Y}_s = \tilde{u}(s, \tilde{X}_s)$ and $Y_s = u(s, X_s)$ imply by Remark 2.2.4 in Chapter 2 that both triples satisfy

$$\sup_{s\in[t_1,T]} \mathbb{E}_{0,\infty}\left[|X_s|^2\right] + \sup_{s\in[t_1,T]} \mathbb{E}_{0,\infty}\left[|Y_s|^2|\right] + \mathbb{E}_{0,\infty}\left[\int_{t_1}^T |Z_s|^2 \,\mathrm{d}s\right] < \infty$$

and coincide. In particular, $\tilde{u}(t_1, x) = \tilde{Y}_{t_1} = Y_{t_1} = u(t_1, x)$.

Strong regularity of u as a Markovian decoupling field to $(\xi, (\mu, \sigma, f))$ follows directly from

- the above argument about uniqueness of X, Y, Z for deterministic initial values and bounded Z,
- and the strong regularity of u as decoupling field to $(\xi, (\mu_H, \sigma_H, f_H))$ for some $H > 0$.

□

Remark 4.2.18. We observe from the proof that the supremum of all $h = T - t$ with t satisfying the hypotheses of Theorem 4.2.17 can be bounded away from 0 by a bound, which only depends on

- $L_{\xi,x}$, $L_{\xi,x} \cdot L_{\sigma,z}$,
- $\|\sigma(\cdot,\cdot,\cdot,0)\|_\infty$, T, $L_{\sigma,z}$,
- the values $(L_H)_{H\in[0,\infty)}$ where L_H is the Lipschitz constant of μ, σ and f on $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times B_H$ w.r.t. to the last 3 components, where $B_H \subset \mathbb{R}^{m\times d}$ denotes the ball of radius H with center 0

and which is monotonically decreasing in these values.

The following natural concept introduces a type of Markovian decoupling field for non-Lipschitz problems (non-Lipschitz in z), to which nevertheless standard Lipschitz results can be applied.

Definition 4.2.19. Let u be a Markovian decoupling field for $(\xi, (\mu, \sigma, f))$.

- We call u *controlled in z* if there exists a constant $C > 0$ such that for all $t_1, t_2 \in [t, T]$, $t_1 \le t_2$, and all initial values X_{t_1}, the corresponding processes (X, Y, Z) from the definition of a Markovian decoupling field satisfy $|Z_s(\omega)| \le C$, for almost all $(s, \omega) \in [t, T] \times \Omega$. If for a fixed triple (t_1, t_2, X_{t_1}) there are different choices for X, Y, Z, then all of them are supposed to satisfy the above control.
- We say that a Markovian decoupling field u on $[t, T]$ is *controlled in z* on a subinterval $[t_1, t_2] \subseteq [t, T]$ if u restricted to $[t_1, t_2]$ is a Markovian decoupling field for $(u(t_2, \cdot), (\mu, \sigma, f))$ that is controlled in z.
- A Markovian decoupling field u on an interval $(s, T]$ is said to be *controlled in z* if it is controlled in z on every compact subinterval $[t, T] \subseteq (s, T]$ with C possibly depending on t.

Remark 4.2.20. Our Markovian decoupling field from Theorem 4.2.17 is obviously controlled in z: Consider (4.7) together with the choice of $t \le t_1$ made in the proof.

Remark 4.2.21. Let $(\xi, (\mu, \sigma, f))$ satisfy MLLC and assume that we have a Markovian decoupling field u on some interval $[t, T]$, which is weakly regular and controlled in z. Then u is also a (standard) decoupling field to a Lipschitz problem obtained through a cutoff as in Theorem 4.2.17. As such it is strongly regular (Theorem 4.2.10) and deterministic (Lemma 4.2.12). But now Lemma 4.2.14 is also applicable, since due to the use of a cutoff we can assume the type of linear growth required there. So, u is also continuous.

Lemma 4.2.22. *Let $(\xi, (\mu, \sigma, f))$ satisfy MLLC. For $0 \le s < t < T$ let u be a weakly regular Markovian decoupling field for $(\xi, (\mu, \sigma, f))$ on $[s, T]$.*
If u is controlled in z on $[s, t]$ and $T - t$ is small enough as required in Theorem 4.2.17 resp. Remark 4.2.18, then u is controlled in z on $[s, T]$.

Proof. Clearly, u is not just controlled in z on $[s, t]$, but also on $[t, T]$ (with a possibly different constant), according to Remark 4.2.20. Define C as the maximum of these two constants.
We only need to control Z by C for the case $s \le t_1 \le t \le t_2 \le T$, the other two cases being trivial.
Now, consider the processes X, Y, Z on the interval $[t_1, t_2]$ corresponding to some initial value X_{t_1} and fulfilling the forward equation, the backward equation and the decoupling condition. Since the restrictions of these processes to $[t_1, t]$ still fulfill these three properties we obtain $|Z_r(\omega)| \le C$ for almost all $r \in [t_1, t], \omega \in \Omega$.
At the same time, if we restrict X, Y, Z to $[t, t_2]$, we observe that these restrictions satisfy the forward equation, backward equation and the decoupling condition for the interval $[t, t_2]$ with X_t as initial value. Therefore, $|Z_r(\omega)| \le C$ holds for a.a. $r \in [t, t_2], \omega \in \Omega$ as well. □

The following important result allows us to connect the MLLC-case to SLC.

Theorem 4.2.23. *Let $(\xi, (\mu, \sigma, f))$ be such that MLLC is satisfied and assume that there exists a weakly regular Markovian decoupling field u to this problem on some interval $[t, T]$. Then u is controlled in z.*

Proof. Let $S \subseteq [t, T]$ be the set of all times $s \in [t, T]$, s.t. u is controlled in z on $[t, s]$.

- Clearly, $t \in S$: For the interval $[t, t] = \{t\}$ one can only choose $t_1 = t_2 = t$ and so $Z : [t, t] \times \Omega \to \mathbb{R}^{m \times d}$ is $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$ - a.e. 0, independently of the initial value X_{t_1}. So, we can take for C any positive value.
- Let $s \in S$ be arbitrary. According to Lemma 4.2.22 there exists an $h > 0$ s.t. u is controlled in z on $[t, (s + h) \wedge T]$, since $\|u((s + h) \wedge T, \cdot)\|_\infty < \infty$ and $L_{u((s+h) \wedge T, \cdot)} < L_{\sigma,z}^{-1}$. Considering Remark 4.2.18 and the requirements $\sup_{s \in [t,T]} \|u(s, \cdot, 0)\|_\infty < \infty$, $L_{u,x} < L_{\sigma,z}^{-1}$, we can choose h independently of s.

This shows $S = [t, T]$ using small interval induction. □

Note that Theorem 4.2.23 implies together with Remark 4.2.21 that a weakly regular Markovian decoupling field to an MLLC problem is deterministic and continuous.

Such a u will be a standard decoupling field to an SLC - problem if we cutoff μ, σ, f appropriately. We can, thereby, extend the whole theory to MLLC - problems:

Theorem 4.2.24. *Let $(\xi, (\mu, \sigma, f))$ satisfy MLLC.*

1. Global uniqueness: *If there are two weakly regular Markovian decoupling fields $u^{(1)}, u^{(2)}$ to this problem on some interval $[t, T]$, then $u^{(1)} = u^{(2)}$ (up to modifications).*
2. Global regularity: *If there exists a weakly regular Markovian decoupling field u to this problem on some interval $[t, T]$, then u is strongly regular.*

Proof. 1. We know that $u^{(1)}$ and $u^{(2)}$ are controlled in z. Choose a passive cutoff (see proof of Theorem 4.2.17) and apply 1. of Theorem 4.2.10.
2. u is controlled in z. Choose a passive cutoff (see proof of Theorem 4.2.17) and apply 2. of Theorem 4.2.10. $\square$

Lemma 4.2.25. *Let $(\xi, \mu, \sigma, f))$ satisfy MLLC and assume that there exists a weakly regular Markovian decoupling field u on some interval $[t, T]$.*
Then for any initial condition $X_t = x \in \mathbb{R}^n$ there is a unique solution (X, Y, Z) of the FBSDE on $[t, T]$ such that

$$\sup_{s\in[t,T]} \mathbb{E}_{0,\infty}[|X_s|^2] + \sup_{s\in[t,T]} \mathbb{E}_{0,\infty}[|Y_s|^2] + \|Z\|_\infty < \infty.$$

Proof. Existence follows from the fact that u is also strongly regular according to 2. of Theorem 4.2.24 and controlled in z according to Theorem 4.2.23.
Uniqueness follows from Corollary 4.2.11: Assume there are two solutions (X, Y, Z) and $(\tilde{X}, \tilde{Y}, \tilde{Z})$ to the FBSDE on $[t, T]$ both satisfying the aforementioned bound. But then they both solve an SLC-conform FBSDE obtained through a passive cutoff, such that u serves as a weakly regular decoupling field for this problem. So, the triples must coincide according to Corollary 4.2.11. $\square$

Definition 4.2.26. Let $I^M_{\max} \subseteq [0, T]$ for $(\xi, (\mu, \sigma, f))$ be the union of all intervals $[t, T] \subseteq [0, T]$ such that there exists a weakly regular Markovian decoupling field u on $[t, T]$.

We will sometimes refer to $I^M_{\max}$ as the *maximal interval*. However, it should not be confused with $I_{\max}$ or $I^b_{\max}$, which are defined in a different way (but fulfill a similar function).

Unfortunately, the maximal interval $I^M_{\max}$ might very well be open to the left. Therefore, we need to make our notions more precise in the following definitions:

Definition 4.2.27. Let $0 \leq t < T$.

- We call a function $u\colon (t, T] \times \mathbb{R}^n \to \mathbb{R}^m$ a Markovian decoupling field for $(\xi, (\mu, \sigma, f))$ on $(t, T]$ if u restricted to $[t', T]$ is a Markovian decoupling field for all $t' \in (t, T]$.
- We call a Markovian decoupling field u on $(t, T]$ *weakly regular* if u restricted to $[t', T]$ is a weakly regular Markovian decoupling field for all $t' \in (t, T]$.
- We call a Markovian decoupling field u on $(t, T]$ *strongly regular* if u restricted to $[t', T]$ is strongly regular for all $t' \in (t, T]$.

Theorem 4.2.28 (Global existence in weak form)**.** *Let $(\xi, (\mu, \sigma, f))$ satisfy MLLC. Then there exists a unique weakly regular Markovian decoupling field u on $I^M_{\max}$. This u is also controlled in z, strongly regular, deterministic and continuous.*
Moreover, either $I^M_{\max} = [0, T]$ or $I^M_{\max} = (t^M_{\min}, T]$, where $0 \leq t^M_{\min} < T$.

Proof. Let $t \in I^M_{\max}$. Obviously there exists a Markovian decoupling field $\check{u}^{(t)}$ on $[t, T]$ satisfying $L_{\check{u}^{(t)},x} < L^{-1}_{\sigma,z}$ and $\sup_{s\in[t,T]} \|\check{u}^{(t)}(s, \cdot, 0)\|_\infty < \infty$. $\check{u}^{(t)}$ is controlled in z and strongly regular due to Theorems 4.2.23 and 4.2.24. We can further assume w.l.o.g. that $\check{u}^{(t)}$ is a continuous function on $[t, T] \times \mathbb{R}^n$ according to Remark 4.2.21. There is only one such $\check{u}^{(t)}$ according to Theorem 4.2.24. Furthermore, for $t, t' \in I^M_{\max}$ the functions $\check{u}^{(t)}$ and $\check{u}^{(t')}$ coincide on $[t \vee t', T]$ because of Theorem 4.2.24.
Define $u(t, \cdot) := \check{u}^{(t)}(t, \cdot)$ for all $t \in I^M_{\max}$. This function u is a Markovian decoupling field on $[t, T]$, since it coincides with $\check{u}^{(t)}$ on $[t, T]$. Therefore, u is a Markovian decoupling field on the whole interval $I^M_{\max}$ and satisfies $L_{u|_{[t,T]},x} < L^{-1}_{\sigma,z}$, $\sup_{s\in[t,T]} \|u|_{[t,T]}(s, 0)\|_\infty < \infty$ for all $t \in I^M_{\max}$. ✓

Uniqueness of u follows directly from Theorem 4.2.24 applied to every interval $[t, T] \subseteq I^M_{\max}$. ✓

Addressing the form of $I^M_{\max}$, we see that $I^M_{\max} = [t,T]$ with $t \in (0,T]$ is not possible: Assume otherwise. According to the above there exists a Markovian decoupling field u on $[t,T]$ s.t. $L_{u,x} < L^{-1}_{\sigma,z}$ and $\sup_{s\in[t,T]} \|u(s,\cdot,0)\|_\infty < \infty$. But then u can be extended a little bit to the left using Theorem 4.2.17 and Lemma 4.2.2, thereby contradicting the definition of $I^M_{\max}$. $\square$

The following result basically states that for a singularity $t^M_{\min}$ to occur u_x has to "explode" at $t^M_{\min}$.

Lemma 4.2.29. *Let $(\xi,(\mu,\sigma,f))$ satisfy MLLC. If $I^M_{\max} = (t^M_{\min},T]$, then*

$$\lim_{t\downarrow t^M_{\min}} L_{u(t,\cdot),x} = L^{-1}_{\sigma,z},$$

where u is the weakly regular Markovian decoupling field according to Theorem 4.2.28.

Proof. We argue indirectly: Clearly, $L_{u(t,\cdot),x} < L^{-1}_{\sigma,z}$ for all $t \in I^M_{\max}$. Assume $\lim_{t\downarrow t^M_{\min}} L_{u(t,\cdot),x} = L^{-1}_{\sigma,z}$ does not hold. Then we can select times $t_n \downarrow t^M_{\min}$, $n \to \infty$ such that

$$\sup_{n\in\mathbb{N}} L_{u(t_n,\cdot),x} < L^{-1}_{\sigma,z}.$$

But then we may choose an $h > 0$ according to Remark 4.2.18 which does not depend on n and then choose n large enough to have $t_n - t^M_{\min} < h$. So, u can be extended to the left to a larger interval $[(t_n - h) \vee 0, T]$ contradicting the definition of $I^M_{\max}$. $\square$

4.3 Solution to the Skorokhod embedding problem

In this section we present a solution to the Skorokhod embedding problem as stated in (SEP) at the beginning of Section 4.1.

4.3.1 Weak solution

Let us now come back to our FBSDE (4.3) associated with the Skorokhod embedding problem, which can be rewritten slightly more generally as

$$\begin{aligned} X^{(1)}_s &= x^{(1)} + \int_t^s 1\,\mathrm{d}W_r, & X^{(2)}_s &= x^{(2)} + \int_t^s Z^2_r\,\mathrm{d}r, \\ Y_s &= g(X^{(1)}_T) - \delta(X^{(2)}_T) - \int_s^T Z_r\,\mathrm{d}W_r, & u(s,X^{(1)}_s,X^{(2)}_s) &= Y_s, \end{aligned} \tag{4.8}$$

for $s \in [t,T]$ and $x = \left(x^{(1)},x^{(2)}\right)^\top \in \mathbb{R}^2$. So, using the notations of Section 4.2 we have

- $d = 1$, $n = 2$, $m = 1$,
- $\mu(t,\omega,x,y,z) = (0,z^2)^\top$,
- $\sigma(t,\omega,x,y,z) = (1,0)^\top$,
- $f(t,\omega,x,y,z) = 0$ and
- $\xi(\omega,x) = g(x^{(1)}) - \delta(x^{(2)})$

for all $(t, \omega, x, y, z) \in [0, T] \times \Omega \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$. In particular, the problem satisfies MLLC if we assume that g and δ are Lipschitz continuous.
Notice that by choosing $x := (x^{(1)}, x^{(2)})^\top := (0,0)^\top$, $t = 0$ and $T = 1$ we will have $X_1^{(1)} = W_1$ and $X_1^{(2)} = \int_0^1 Z_s^2 \, \mathrm{d}s$, which makes the FBSDE equivalent to (4.3).
With the general results of Section 4.2.2 at hand we are capable to solve this system of equations. In other words, we are able to perform the **second step** of our algorithm to solve to SEP.

Lemma 4.3.1. *Assume δ and g are Lipschitz continuous. Then for the FBSDE* (4.8) *there exists a unique weakly regular Markovian decoupling field u on $[0, T]$. This u is strongly regular, controlled in z, deterministic and continuous.*
In particular, equation (4.3) *has a unique solution (Y, Z) s.t. $\|Z\|_\infty < \infty$.*

Proof. Using Theorem 4.2.28 we know that there exists a unique weakly regular Markovian decoupling field u on $I_{\max}^M$. This u is furthermore strongly regular, controlled in z, deterministic and continuous. It remains to prove $I_{\max}^M = [0, T]$. Due to Lemma 4.2.29 it is sufficient to show the existence of a constant $C \in [t, \infty)$ such that $L_{u(t,\cdot),x} \leq C < L_{\sigma,z}^{-1}$ for all $t \in I_{\max}^M$. In our case $L_{\sigma,z}^{-1} = \infty$, so we have to prove that the weak partial derivatives of u with respect to $x^{(1)}$ and $x^{(2)}$ are both uniformly bounded.
Now, fix $t \in I_{\max}^M$ and consider the corresponding FBSDE on $[t, T]$: First notice that the associated (X, Y, Z) depend on the initial value $x = (x^{(1)}, x^{(2)})^\top \in \mathbb{R}^2$ and are even weakly differentiable with respect to the initial value x according to the strong regularity of u. For more on rules regarding working with weak derivatives consult Section 2.1.2 of Chapter 2.
Let us first look at the matrix $\frac{\mathrm{d}}{\mathrm{d}x} X$. We have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x^{(2)}} X_s^{(1)} &= 0, \\
\frac{\mathrm{d}}{\mathrm{d}x^{(2)}} X_s^{(2)} &= 1 + \int_t^s 2 Z_r \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} Z_r \, \mathrm{d}r, \\
\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} X_s^{(1)} &= 1, \\
\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} X_s^{(2)} &= \int_t^s 2 Z_r \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} Z_r \, \mathrm{d}r,
\end{aligned}
$$

a.s. for $s \in [t, T]$, for almost all $x = (x^{(1)}, x^{(2)})^\top \in \mathbb{R}^2$. In particular, the 2×2 - matrix $\frac{\mathrm{d}}{\mathrm{d}x} X_s$ is invertible if and only if $\frac{\mathrm{d}}{\mathrm{d}x^{(2)}} X_s^{(2)}$ is not 0. We will later see that it remains positive on the whole interval allowing us to apply the chain rule of Lemma A.3.1 in order to write $\frac{\mathrm{d}}{\mathrm{d}x} u(s, X_s) \frac{\mathrm{d}}{\mathrm{d}x} X_s$. But let us first proceed by differentiating the backward equation in (4.8) with respect to $x^{(2)}$:

$$
\frac{\mathrm{d}}{\mathrm{d}x^{(2)}} Y_s = -\delta'(X_T^{(2)}) \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} X_T^{(2)} - \int_s^T \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} Z_r \, \mathrm{d}W_r.
$$

To be precise the above holds a.s. for every $s \in [t, T]$, for almost all $x = (x^{(1)}, x^{(2)})^\top \in \mathbb{R}^2$.
We can assume without loss of generality that $\frac{\mathrm{d}}{\mathrm{d}x} X$, $\frac{\mathrm{d}}{\mathrm{d}x^{(2)}} Y$ are continuous in time.
Now, define a stopping time τ via

$$
\tau := \inf \left\{ s \in [t, T] \;\middle|\; \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} X_s^{(2)} \leq 0 \right\} \wedge T
$$

For $s \in [t, \tau)$ we have $\frac{\mathrm{d}}{\mathrm{d}x} u(s, X_s) \frac{\mathrm{d}}{\mathrm{d}x} X_s$ according to the chain rule of Lemma A.3.1 and in particular $\frac{\mathrm{d}}{\mathrm{d}x^{(2)}} u(s, X_s^{(1)}, X_s^{(2)}) \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} X_s^{(2)} = \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} Y_s$. Now, let us define

$$
V_s := \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} u(s, X_s^{(1)}, X_s^{(2)}), \; s \in [t, T] \quad \text{and} \quad \tilde{Z}_r := \frac{\frac{\mathrm{d}}{\mathrm{d}x^{(2)}} Z_r}{\frac{\mathrm{d}}{\mathrm{d}x^{(2)}} X_r^{(2)}} \mathbf{1}_{\{r \in [t, \tau)\}},
$$

so the dynamics of $\left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_s^{(2)}\right)^{-1}$ can be written as

$$\left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_{s\wedge\tilde{\tau}}^{(2)}\right)^{-1} = 1 - \int_t^{s\wedge\tilde{\tau}} 2Z_r\tilde{Z}_r\left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_r^{(2)}\right)^{-1}\mathrm{d}r, \tag{4.9}$$

for an arbitrary stopping time $\tilde{\tau} < \tau$ with values in $[t, T]$. We also have $\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}Y_s = V_s\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_s^{(2)}$ and, therefore,

$$V_s = \frac{\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}Y_s}{\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_s^{(2)}},$$

for $s \in [t, \tau)$ and more generally for every s s.t. $\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_s^{(2)} > 0$. In particular, V_s has a continuous modification on $[t, \tau)$.
Applying Itô's formula and using dynamics of $\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}Y$ and $\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X^{(2)}$ we easily obtain the dynamics of $V_{s\wedge\tilde{\tau}}$:

$$\begin{aligned} V_{s\wedge\tilde{\tau}} &= V_t + \int_t^{s\wedge\tilde{\tau}} -2Z_r\tilde{Z}_r\left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_r^{(2)}\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}Y_r\,\mathrm{d}r + \int_t^{s\wedge\tilde{\tau}}\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}Z_r\left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_r^{(2)}\right)^{-1}\mathrm{d}W_r \\ &= V_t + \int_t^{s\wedge\tilde{\tau}}(-2Z_rV_r)\tilde{Z}_r\,\mathrm{d}r + \int_t^{s\wedge\tilde{\tau}}\tilde{Z}_r\,\mathrm{d}W_r \end{aligned} \tag{4.10}$$

for any stopping time $\tilde{\tau} < \tau$ with values in $[t, T]$.
Note that V and $(-2ZV)$ are bounded processes: Z is bounded because we work with Markovian decoupling fields and V is bounded due to its definition. Therefore, $\tilde{Z}\mathbf{1}_{[\cdot\leq\tilde{\tau}]}$ is in $BMO(\mathbb{P})$ according to Theorem A.1.11 with a $BMO(\mathbb{P})$ - norm which does not depend on $\tilde{\tau} < \tau$ and so in particular $\mathbb{E}[\int_t^\tau |2Z_r\tilde{Z}_r|^2\,\mathrm{d}r] < \infty$. From (4.9) we can actually deduce that $\tau = T$ must hold almost surely: Indeed, (4.9) implies that

$$\left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_{s\wedge\tilde{\tau}}^{(2)}\right)^{-1} = \exp\left(-\int_t^{s\wedge\tilde{\tau}}2Z_r\tilde{Z}_r\,\mathrm{d}r\right)$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_{s\wedge\tilde{\tau}}^{(2)} = \exp\left(\int_t^{s\wedge\tilde{\tau}}2Z_r\tilde{Z}_r\,\mathrm{d}r\right)$$

for all stopping times $\tilde{\tau} < \tau$ with values in $[t, T]$. This implies, using continuity of $s \mapsto \frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_s^{(2)}$,

$$\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_\tau^{(2)} = \exp\left(\int_t^\tau 2Z_r\tilde{Z}_r\,\mathrm{d}r\right) > 0,$$

which gives us $\tau = T$ a.s. because $\{\tau < T\} \subset \left\{\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_\tau^{(2)} = 0\right\}$, due to continuity of $\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X^{(2)}$.
So, we have that $\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X^{(2)}$ is positive on the whole $[t, T]$ and therefore $\frac{\mathrm{d}}{\mathrm{d}x}X$ is invertible on $[t, T]$.
Setting $\tilde{W}_s := W_s - \int_t^s 2Z_rV_r\,\mathrm{d}r$, $s \in [t, T]$ we can reformulate (4.10) to

$$V_s = V_t + \int_t^s \tilde{Z}_r\,\mathrm{d}\tilde{W}_r.$$

This means that V_s can be viewed as the conditional expectation of

$$V_T = \frac{\mathrm{d}}{\mathrm{d}x^{(2)}}u(T, X_T^{(1)}, X_T^{(2)}) = -\delta'(X_T^{(2)})$$

with respect to $\mathcal{F}_s$ and some probability measure, which turns $\tilde{W}$ into a Brownian motion on $[t,T]$. Remember here that $2Z_rV_r$ is bounded on $[t,T]$. Hence, we conclude that V_t and therefore $\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}u(t,x^{(1)},x^{(2)})$ is bounded by $\|\delta'\|_\infty$ for almost all $x=(x^{(1)},x^{(2)})^\top\in\mathbb{R}^2$. This value is independent of t. ✓

Secondly, we have to bound $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u(t,x^{(1)},x^{(2)})$. To this end we differentiate the equations in (4.8) with respect to $x^{(1)}$:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_s^{(1)}=1,\\
&\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_s^{(2)}=\int_t^s 2Z_r\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}Z_r\,\mathrm{d}r,\\
&\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}Y_s=g'(X_T^{(1)})-\delta'(X_T^{(2)})\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_T^{(2)}-\int_s^T\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}Z_r\,\mathrm{d}W_r,\\
&\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u(s,X_s^{(1)},X_s^{(2)})+\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}u(s,X_s^{(1)},X_s^{(2)})\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_s^{(2)}=\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}Y_s,
\end{aligned}
$$

and define

$$
U_s:=\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u(s,X_s^{(1)},X_s^{(2)}),\quad \check{Z}_r:=\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}Z_r-\tilde{Z}_r\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_r^{(2)}.
$$

Note that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_s^{(2)}&=\int_t^s 2Z_r\left(\check{Z}_r+\tilde{Z}_r\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_r^{(2)}\right)\mathrm{d}r,\\
U_s&=\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}Y_s-V_s\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_s^{(2)},\quad \text{a.s. for } s\in[t,T],
\end{aligned}
$$

which allows us to deduce the dynamics of U from dynamics of $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}Y$, $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X^{(2)}$ and V using Itô formula:

$$
\begin{aligned}
U_s=U_t&+\int_t^s 1\,\mathrm{d}\Big(\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}Y_r\Big)-\int_t^s V_r\,\mathrm{d}\Big(\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_r^{(2)}\Big)-\int_t^s\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_r^{(2)}\,\mathrm{d}V_r\\
&=U_t+\int_t^s\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}Z_r\,\mathrm{d}W_r-2\int_t^s V_rZ_r\left(\check{Z}_r+\tilde{Z}_r\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_r^{(2)}\right)\mathrm{d}r\\
&\qquad-\int_t^s\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_r^{(2)}\left(-2Z_rV_r\tilde{Z}_r\,\mathrm{d}r+\tilde{Z}_r\,\mathrm{d}W_r\right)
\end{aligned}
$$

where the marked terms either merge into one or cancel out and the equation simplifies to

$$
\begin{aligned}
U_s=&U_t+\int_t^s(-2Z_rV_r\check{Z}_r)\,\mathrm{d}r+\int_t^s\check{Z}_r\,\mathrm{d}W_r.\\
=&U_t+\int_t^s\check{Z}_r\,\mathrm{d}\tilde{W}_r.
\end{aligned}
\tag{4.11}
$$

By the same argument as for the process V we deduce that U and therefore $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u(t,x^{(1)},x^{(2)})$ is bounded by $\|g'\|_\infty=L_g$ for almost all $x^{(1)},x^{(2)}$, where L_g is the Lipschitz constant of g, i.e. the infimum of all Lipschitz constants (see also Lemma 2.1.4).
This shows $I_{\max}^M=[0,T]$. ✓

Finally, Lemma 4.2.25 shows that there is a unique solution (X,Y,Z) to the FBSDE on $[0,T]$ for any initial value $(X_0^{(1)},X_0^{(2)})^\top=(x^{(1)},x^{(2)})^\top\in\mathbb{R}^2$ such that

$$
\sup_{s\in[0,T]}\mathbb{E}_{0,\infty}[|X_s|^2]+\sup_{s\in[0,T]}\mathbb{E}_{0,\infty}[|Y_s|^2]+\|Z\|_\infty<\infty,
$$

which is equivalent to the simpler condition $\|Z\|_\infty < \infty$ as we claim:

If $\|Z\|_\infty < \infty$, then according to the forward equation

$$\|X^{(2)}\|_\infty \leq |x^{(2)}| + T\|Z\|_\infty^2 < \infty,$$

$$\sup_{s\in[0,T]} \mathbb{E}_{0,\infty}[|X_s^{(1)}|^2] = |x^{(1)}|^2 + \sup_{s\in[0,T]} \mathbb{E}_{0,\infty}[|W_s|^2] = |x^{(1)}|^2 + T < \infty$$

and according to the backward equation together with the Jensen and Minkowski inequalities

$$\begin{aligned}
\left(\sup_{s\in[0,T]} \mathbb{E}_{0,\infty}[|Y_s|^2]\right)^{\frac{1}{2}} &= \left(\sup_{s\in[0,T]} \mathbb{E}_{0,\infty}\left[\left|\mathbb{E}\left[g(X_T^{(1)}) - \delta(X_T^{(2)})\middle|\mathcal{F}_s\right]\right|^2\right]\right)^{\frac{1}{2}} \\
&\leq \left(\mathbb{E}_{0,\infty}\left[\left|g(X_T^{(1)}) - \delta(X_T^{(2)})\right|^2\right]\right)^{\frac{1}{2}} \leq \left(\mathbb{E}_{0,\infty}\left[\left|g(X_T^{(1)})\right|^2\right]\right)^{\frac{1}{2}} + \left(\mathbb{E}_{0,\infty}\left[\left|\delta(X_T^{(2)})\right|^2\right]\right)^{\frac{1}{2}} \\
&\leq |g(0)| + L_g\left(\mathbb{E}_{0,\infty}\left[\left|X_T^{(1)}\right|^2\right]\right)^{\frac{1}{2}} + |\delta(0)| + L_\delta\left(\mathbb{E}_{0,\infty}\left[\left|X_T^{(2)}\right|^2\right]\right)^{\frac{1}{2}} < \infty,
\end{aligned}$$

where L_g, L_δ are Lipschitz constants of g, δ. □

For the following result we use the notations of Section 4.1. As before we assume that β is bounded away from 0. Under this condition H^{-1} is well-defined and Lipschitz continuous. Therefore, $\delta = \hat{\delta} \circ H^{-1}$ is Lipschitz continuous if $\hat{\delta}$ is Lipschitz continuous, which is equivalent to α is being bounded.

Lemma 4.3.2. *Suppose g and δ are both Lipschitz continuous with Lipschitz constants L_g, L_δ. Then there exist a Brownian motion B, a random time $\tilde{\tau} \leq H^{-1}(L_g^2)$ and a constant $c \in \mathbb{R}$ such that $c + \int_0^{\tilde{\tau}} \alpha_s \,\mathrm{d}s + \int_0^{\tilde{\tau}} \beta_s \,\mathrm{d}B_s$ has law ν.*

Proof. First we solve FBSDE (4.3) using Lemma 4.3.1 such that the corresponding Z is bounded. According to Lemma 4.3.5, which we prove a bit later, we can even assume that Z is bounded by L_g. Now, we set $c := Y_0$ and construct B and $\tilde{\tau}$ as in the proof of Lemma 4.1.2.
Moreover, $\tilde{\tau} = H^{-1}\left(\int_0^1 Z_s^2 \,\mathrm{d}s\right)$ is bounded by $H^{-1}(L_g^2)$ since Z is bounded by L_g and H^{-1} is increasing. □

Remark 4.3.3. It is a priori not clear that the random time $\tilde{\tau}$ is actually a stopping time with respect to the filtration

$$\left(\mathcal{F}_s^B\right)_{s\in[0,\infty)} := \left(\sigma\left(\mathcal{F}_0, (B_r)_{r\in[0,s]}\right)_{s\in[0,\infty)}\right)$$

as also mentioned in Remark 1.2 in [AHI08]. However, we will show a sufficient criterion for this in terms of regularity properties of the Markovian decoupling field u.

Remark 4.3.4. The boundedness of the stopping time solving the Skorokhod embedding problem has not been investigated so frequently. However, very recently it gained attention in [AS11] and [AHS13]. It particular, its economic interest comes from its natural applications in the context of game theory (see [SS09]).

4.3.2 Strong solution

This subsection is devoted to preforming the **fourth step** of our algorithm and finally to solve the Skorokhod embedding problem in the strong sense. Our main goal is to show that if g, δ are sufficiently smooth, then $\tilde{\tau}$ and B constructed so far will have the property that $\tilde{\tau}$ is indeed a stopping time w.r.t.

the filtration $\left(\mathcal{F}_s^B\right)_{s\in[0,\infty)}$ generated by the Brownian motion B. More precisely, we will assume that g and δ are three times weakly differentiable with bounded derivatives. We will also use that g is non-decreasing and not constant. The line of argument will be based on a rather deep analysis of regularity properties of the decoupling field u.

First let us prove the following very useful result about the solution (Y,Z) to FBSDE (4.3) constructed in Lemma 4.3.1:

Lemma 4.3.5. *Assume that δ and g are Lipschitz continuous. Let u be the unique weakly regular Markovian decoupling field to the problem (4.8) on $[0,T]$ constructed in Lemma 4.3.1. Then for any $t\in[0,T)$ and initial condition $(X_t^{(1)},X_t^{(2)})^\top=(x^{(1)},x^{(2)})^\top\in\mathbb{R}^2$ the associated process Z on $[t,T]$ satisfies $\|Z\|_\infty\le L_g=\|g'\|_\infty$.*
Furthermore, if the weak derivative $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u$ has a version whose restriction to the set $[t,T)\times\mathbb{R}^2$ is continuous in the first two components $(s,x^{(1)})$, then

$$Z_s(\omega)=\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u\left(s,X_s^{(1)}(\omega),X_s^{(2)}(\omega)\right)$$

for almost all $(s,\omega)\in[t,T]\times\Omega$.

Proof. We already know that Z is bounded according to Lemma 4.3.1 but not in the form of the more explicit bound $\|Z\|_\infty\le L_g$.
Notice that $\lim_{h\downarrow 0}\frac{1}{h}\int_s^{s+h}Z_r(\omega)\,\mathrm{d}r=Z_s(\omega)$ for almost all $(\omega,s)\in\Omega\times[t,T)$ due to the fundamental Theorem of Lebesgue integral calculus.
Now, take some $s\in[t,T)$ s.t. $\lim_{h\downarrow 0}\frac{1}{h}\int_s^{s+h}Z_r\,\mathrm{d}r=Z_s$ almost surely. Almost all $s\in[t,T)$ have this property. Choose any $h>0$ s.t $s+h<T$ and consider the expression

$$\frac{1}{h}\mathbb{E}[Y_{s+h}(W_{s+h}-W_s)|\mathcal{F}_s]$$

for small $h>0$. On the one hand we can write using Itô's formula:

$$Y_{s+h}(W_{s+h}-W_s)=\int_s^{s+h}Y_r\,\mathrm{d}W_r+\int_s^{s+h}(W_r-W_s)Z_r\,\mathrm{d}W_r+\int_s^{s+h}Z_r\,\mathrm{d}r,$$

which leads to

$$\frac{1}{h}\mathbb{E}[Y_{s+h}(W_{s+h}-W_s)|\mathcal{F}_s]=\frac{1}{h}\mathbb{E}\left[\int_s^{s+h}Z_r\,\mathrm{d}r\middle|\mathcal{F}_s\right]\to Z_s\quad\text{for}\quad h\to 0.$$

On the other hand we can use the decoupling condition to write

$$\begin{aligned}Y_{s+h}(W_{s+h}-W_s)=&u\left(s+h,X_{s+h}^{(1)},X_{s+h}^{(2)}\right)(W_{s+h}-W_s)\\=&u\left(s+h,X_{s+h}^{(1)},X_s^{(2)}\right)(W_{s+h}-W_s)\\&+\left(u\left(s+h,X_{s+h}^{(1)},X_{s+h}^{(2)}\right)-u\left(s+h,X_{s+h}^{(1)},X_s^{(2)}\right)\right)(W_{s+h}-W_s).\end{aligned}$$

After applying conditional expectations to the above equation we investigate the two summands on the right hand side separately:
First summand: Remember:

- $X_s^{(1)},X_s^{(2)}$ are $\mathcal{F}_s$ measurable,
- $X_{s+h}^{(1)}=X_s^{(1)}+(W_{s+h}-W_s)$,

- $W_{s+h} - W_s$ is independent of $\mathcal{F}_s$,
- u is deterministic, i.e. can be assumed to be a function of $\left(s, x^{(1)}, x^{(2)}\right) \in [0,T] \times \mathbb{R} \times \mathbb{R}$ only.

These properties imply using integration by parts

$$\begin{aligned}
\mathbb{E}\Big[u\left(s+h, X^{(1)}_{s+h}, X^{(2)}_s\right)(W_{s+h} - W_s)\Big|\mathcal{F}_s\Big]& \\
&= \int_{\mathbb{R}} u\left(s+h, X^{(1)}_s + z\sqrt{h}, X^{(2)}_s\right) z\sqrt{h}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\,\mathrm{d}z \\
&= \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u\left(s+h, X^{(1)}_s + z\sqrt{h}, X^{(2)}_s\right) h\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\,\mathrm{d}z,
\end{aligned}$$

which means

$$\lim_{h\downarrow 0}\frac{1}{h}\mathbb{E}\left[u\left(s+h, X^{(1)}_{s+h}, X^{(2)}_s\right)(W_{s+h} - W_s)\Big|\mathcal{F}_s\right] = \frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u\left(s, X^{(1)}_s, X^{(2)}_s\right),$$

if $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u$ is continuous in the first two components on $[0,T) \times \mathbb{R}^2$. Here we use that $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u$ is bounded by $\|g'\|_\infty$ according to the proof of Lemma 4.3.1. But even if $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u$ is not continuous in the first two components, we can still at least control the value

$$\left|\frac{1}{h}\mathbb{E}\left[u\left(s+h, X^{(1)}_{s+h}, X^{(2)}_s\right)(W_{s+h} - W_s)\Big|\mathcal{F}_s\right]\right|$$

by $\|g'\|_\infty$.

Second summand: Remember:

- u is also Lipschitz continuous in the last component and we can use $\|\delta'\|_\infty$ as a Lipschitz constant,
- $X^{(2)}_{s+h} = X^{(2)}_s + \int_s^{s+h} Z_r^2\,\mathrm{d}r$.

These properties allow us to estimate

$$\begin{aligned}
&\frac{1}{h}\left|\mathbb{E}\left[\left(u\left(s+h, X^{(1)}_{s+h}, X^{(2)}_{s+h}\right) - u\left(s+h, X^{(1)}_{s+h}, X^{(2)}_s\right)\right)(W_{s+h} - W_s)\Big|\mathcal{F}_s\right]\right| \\
&\quad\le \frac{1}{h}\mathbb{E}\left[\left|u\left(s+h, X^{(1)}_{s+h}, X^{(2)}_{s+h}\right) - u\left(s+h, X^{(1)}_{s+h}, X^{(2)}_s\right)\right| \cdot |W_{s+h} - W_s|\Big|\mathcal{F}_s\right] \\
&\quad\le \frac{1}{h}\mathbb{E}\left[\|\delta'\|_\infty\left(\int_s^{s+h} Z_r^2\,\mathrm{d}r\right) \cdot |W_{s+h} - W_s|\Big|\mathcal{F}_s\right] \le \frac{1}{h}\|\delta'\|_\infty h\|Z\|_\infty^2\mathbb{E}[|W_{s+h} - W_s|],
\end{aligned}$$

which clearly goes to 0 as $h \to 0$.

Conclusion: We have shown

$$Z_s = \lim_{h\downarrow 0}\frac{1}{h}\mathbb{E}[Y_{s+h}(W_{s+h} - W_s)|\mathcal{F}_s] = \lim_{h\downarrow 0}\frac{1}{h}\mathbb{E}\left[u\left(s+h, X^{(1)}_{s+h}, X^{(2)}_s\right)(W_{s+h} - W_s)\Big|\mathcal{F}_s\right],$$

which is equal to $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u\left(s, X^{(1)}_s, X^{(2)}_s\right)$ a.s. if $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u$ is continuous in the first two components on $[0,T) \times \mathbb{R}^2$ and bounded by $\|g'\|_\infty$ otherwise. □

For the following let u be the unique weakly regular Markovian decoupling field to the problem (4.8) constructed in Lemma 4.3.1, where at least for the following result we assume $T = 1$. We also use definitions and notations from the proof of Lemma 4.1.2.

Theorem 4.3.6. *Assume that* $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u$ *is*

- *Lipschitz continuous in the first two components on compact subsets of* $[0,1)\times\mathbb{R}^2$,
- $\mathbb{R}\backslash\{0\}$ *- valued on* $[0,1)\times\mathbb{R}^2$.

Then $\tilde{\tau}$ *is a stopping time with respect to the filtration* $(\mathcal{F}^B_\cdot) = (\mathcal{F}^B_s)_{s\in[0,\infty)}$.

Proof. We consider the system (4.8) for $t=0$ and $x^{(1)} = x^{(2)} = 0$. According to Lemma 4.3.5 we can assume

$$Z = \frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u\left(\cdot, X^{(1)}_\cdot, X^{(2)}_\cdot\right)$$

and, thereby, have

$$X^{(2)}_s = \int_0^s Z_r^2\,\mathrm{d}r = \int_0^s \left(\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u\left(r, X^{(1)}_r, X^{(2)}_r\right)\right)^2\mathrm{d}r$$

for all $s\in[0,T]$. So, we can assume that $X^{(1)}$

- is Lipschitz continuous and strictly increasing in s due to positivity of $\left(\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u\right)^2$ on $[0,1)\times\mathbb{R}^2$,
- starts in 0.

Therefore, for every $\omega\in\Omega$ the path

$$H^{-1}\left(X^{(2)}_\cdot(\omega)\right) : [0,1]\to[0,\infty)$$

is also Lipschitz continuous and strictly increasing in time and, therefore, has a continuous and strictly increasing inverse function on the interval

$$\left[0, H^{-1}\left(X^{(2)}_1(\omega)\right)\right] = [0,\tilde{\tau}(\omega)].$$

It is straightforward to see that this inverse is given by the process σ from the proof of Lemma 4.1.2. We can now calculate the weak derivative of σ: Firstly, note $\left(H^{-1}\right)'(x) = \frac{1}{H'(H^{-1}(x))}$ and also $H^{-1}(X^{(2)}_{\sigma_r}(\omega)) = r$ or equivalently $X^{(2)}_{\sigma_r}(\omega) = H(r)$. So, we can calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\sigma_r = \frac{1}{\frac{\mathrm{d}}{\mathrm{d}s}\left(H^{-1}\left(X^{(2)}_s\right)\right)\Big|_{s=\sigma_r}} &= \frac{1}{(H^{-1})'\left(X^{(2)}_{\sigma_r}\right)Z^2_{\sigma_r}}\\ &= \frac{H'(r)}{\left(\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u\left(\sigma_r, X^{(1)}_{\sigma_r}, X^{(2)}_{\sigma_r}\right)\right)^2} = \frac{\beta_r^2}{\left(\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u\right)^2(\sigma_r, W_{\sigma_r}, H(r))}\end{aligned} \tag{4.12}$$

on $\{\sigma_r<1\}$. Observe at this point that

$$\{\sigma_r<1\} = \left\{r < H^{-1}\left(X^{(2)}_1\right)\right\} = \{r<\tilde{\tau}\}.$$

If we define $\sigma_r := 1$ for $r>\tilde{\tau}$, then σ is still continuous and we have

$$\tilde{\tau} = \inf\left\{r\in[0,\infty)\,|\,\sigma_r\geq 1\right\}.$$

It is also straightforward to see $Z_{\sigma_r} = \frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u\left(\sigma_r, W_{\sigma_r}, H(r)\right)$ for $r\in[0,\tilde{\tau})$.
Now, remember $B_r = \int_0^r \frac{1}{\beta_s}\,\mathrm{d}Y_{\sigma_s}$ for $r\in[0,\tilde{\tau}]$ and also $Y_s - Y_0 = \int_0^s Z_r\,\mathrm{d}W_r$ for $s\in[0,1]$, so

$$\int_0^r \frac{\beta_s}{Z_{\sigma_s}}\,\mathrm{d}B_s = \int_0^r \frac{\beta_s}{Z_{\sigma_s}}\frac{1}{\beta_s}\,\mathrm{d}Y_{\sigma_s} = \int_0^r \frac{1}{Z_{\sigma_s}}Z_{\sigma_s}\,\mathrm{d}W_{\sigma_s} = W_{\sigma_r}.$$

So, if we define $\Sigma_r := W_{\sigma_r}$, we obtain:

$$\Sigma_r = \int_0^r \frac{\beta_s}{\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u\left(\sigma_s, \Sigma_s, H(s)\right)} \mathrm{d}B_s,$$

for $r \in [0, \tilde{\tau})$. So, to sum up σ, Σ fulfill on $[0, \tilde{\tau})$ the dynamics

$$\begin{aligned}
\sigma_r &= 0 + \int_0^r \frac{\beta_s^2}{\left(\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u\right)^2 (\sigma_s, \Sigma_s, H(s))} \mathrm{d}s + \int_0^r 0 \, \mathrm{d}B_s, \\
\Sigma_r &= 0 + \int_0^r 0 \, \mathrm{d}s + \int_0^r \frac{\beta_s}{\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u\left(\sigma_s, \Sigma_s, H(s)\right)} \mathrm{d}B_s,
\end{aligned}$$

where $r \in [0, \tilde{\tau})$. Note that the coefficients of this dynamical system are locally Lipschitz continuous in (σ, Σ).
Now, for any $K_1, K_2 > 0$ and $K_3 \in (0, 1)$ define a bounded random variable τ_{K_1,K_2,K_3} via

$$\tau_{K_1,K_2,K_3} := K_1 \wedge \inf\left\{r \in [0, \infty) \,|\, |\Sigma_r| \geq K_2\right\} \wedge \inf\left\{r \in [0, \infty) \,|\, \sigma_r \geq K_3\right\}.$$

Note that σ and Σ both remain bounded on $[0, \tau_{K_1,K_2,K_3}]$. Therefore, on $[0, \tau_{K_1,K_2,K_3}]$ the pair (σ, Σ) coincides with the unique solution $(\sigma^{K_1,K_2,K_3}, \Sigma^{K_1,K_2,K_3})$ to a Lipschitz problem, which is automatically progressively measurable w.r.t. the filtration $(\mathcal{F}^B_\cdot)$. Note that

$$\tau_{K_1,K_2,K_3} = K_1 \wedge \inf\left\{r \in [0, \infty \,\middle|\, |\Sigma_r^{K_1,K_2,K_3}| \geq K_2\right\} \wedge \inf\left\{r \in [0, \infty) \,\middle|\, \sigma_r^{K_1,K_2,K_3} \geq K_3\right\},$$

which is clearly a stopping time w.r.t. $(\mathcal{F}^B_\cdot)$. Furthermore, due to continuity of Σ and σ

$$\tilde{\tau} = \sup_{K_3 \in (0,1), K_1, K_2 > 0} \tau_{K_1,K_2,K_3},$$

which makes it a stopping time w.r.t. $(\mathcal{F}^B_\cdot)$ as well. □

In order to deduce sufficient conditions for Theorem 4.3.6 to hold we need to investigate higher order derivatives of u:

Assume that g, δ, g' and δ' are Lipschitz continuous and consider the following MLLC system with $d = 1$, $n = 2$ and $m = 3$:

$$\begin{aligned}
X_s^{(1)} &= x^{(1)} + \int_t^s 1 \, \mathrm{d}W_r, \qquad X_s^{(2)} = x^{(2)} + \int_t^s \left(Z_r^{(0)}\right)^2 \mathrm{d}r, \\
Y_s^{(0)} &= g(X_T^{(1)}) - \delta(X_T^{(2)}) - \int_s^T Z_r^{(0)} \, \mathrm{d}W_r, && u^{(0)}(s, X_s^{(1)}, X_s^{(2)}) = Y_s^{(0)}, \\
Y_s^{(1)} &= g'(X_T^{(1)}) - \int_s^T Z_r^{(1)} \, \mathrm{d}W_r - \int_s^T \left(-2Z_r^{(0)} Y_r^{(2)}\right) Z_r^{(1)} \, \mathrm{d}r, && u^{(1)}(s, X_s^{(1)}, X_s^{(2)}) = Y_s^{(1)}, \\
Y_s^{(2)} &= -\delta'(X_T^{(2)}) - \int_s^T Z_r^{(2)} \, \mathrm{d}W_r - \int_s^T \left(-2Z_r^{(0)} Y_r^{(2)}\right) Z_r^{(2)} \, \mathrm{d}r, && u^{(2)}(s, X_s^{(1)}, X_s^{(2)}) = Y_s^{(2)}.
\end{aligned} \tag{4.13}$$

Theorem 4.3.7. *For the above problem* (4.13) *we have* $I_{\max}^M = [0, T]$. *Furthermore,*

$$u^{(0)} = u, \quad u^{(1)} = \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u \quad \textit{and} \quad u^{(2)} = \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} u,$$

a.e. where u is the unique weakly regular Markovian decoupling field to the problem (4.8).
In particular, u is twice weakly differentiable w.r.t. x with uniformly bounded derivatives.

Proof. The proof is in parts akin to the proof of Lemma 4.3.1 and we will seek to keep these parts short.
Let $u^{(i)}$, $i = 0, 1, 2$ be the unique weakly regular Markovian decoupling field on $I^M_{\max}$. We can assume $u^{(i)}$ to be continuous functions on $I^M_{\max} \times \mathbb{R}^2$ (Theorem 4.2.28).
Let $t \in I^M_{\max}$. For an arbitrary initial condition $x \in \mathbb{R}^2$ consider the corresponding processes

$$X^{(1)}, X^{(2)}, Y^{(0)}, Y^{(1)}, Y^{(2)}, Z^{(0)}, Z^{(1)}, Z^{(2)}$$

on $[t, T]$. Note that $X^{(1)}, X^{(2)}, Y^{(0)}, Z^{(0)}$ solve FBSDE (4.8), which implies that they coincide with the processes $X^{(1)}, X^{(2)}, Y, Z$ from (4.8) if we assume

$$\sum_{i=1}^{2} \sup_{s\in[t,T]} \mathbb{E}_{0,\infty}[|X^{(i)}_s|^2] + \sup_{s\in[t,T]} \mathbb{E}_{0,\infty}[|Y_s|^2] + \|Z\|_\infty + \sum_{i=0}^{2} \sup_{s\in[t,T]} \mathbb{E}_{0,\infty}[|Y^{(i)}_s|^2] + \sum_{i=0}^{2} \|Z^i\|_\infty < \infty,$$

according to Lemma 4.2.25. This condition is fulfilled due to strong regularity and the fact that we work with Markovian decoupling fields.
Now, $Y^{(0)} = Y$ implies $u(t, x) = u^{(0)}(t, x)$ for all $t \in I^M_{\max}, x \in \mathbb{R}^2$, where $I^M_{\max}$ is the maximal interval for the problem given by (4.13).

We now claim that $Y^{(1)}, Y^{(2)}$ are bounded processes: Using the backward equation we have

$$Y^{(2)}_s = \mathbb{E}_s\left[-\delta'(X^{(2)}_T)\right] - \mathbb{E}_s\left[\int_s^T \left(-2Z^{(0)}_r Y^{(2)}_r\right) Z^{(2)}_r \,\mathrm{d}r\right]$$

and, therefore,

$$|Y^{(2)}_s| \le \|\delta'\|_\infty + \int_s^T 2\|Z^{(0)}\|_\infty \|Z^{(2)}\|_\infty \mathbb{E}_s\left[\left|Y^{(2)}_r\right|\right] \mathrm{d}r,$$

for $s \in [t, T]$, which using Gronwall's lemma implies

$$|Y^{(2)}_s| = \mathbb{E}_s\left[\left|Y^{(2)}_s\right|\right] \le \|\delta'\|_\infty \exp\left(2T\|Z^{(0)}\|_\infty \|Z^{(2)}\|_\infty\right).$$

This in turn automatically implies boundedness of $Y^{(1)}$ according to its dynamics. ✓

Furthermore, $Y^{(1)}, Z^{(1)}$ and $Y^{(2)}, Z^{(2)}$ satisfy the BSDE which is also fulfilled by the processes $U, \check{Z}$ and $V, \tilde{Z}$ from the proof of Lemma 4.3.1 (see (4.10) and (4.11)) and so in particular

$$\begin{aligned} Y^{(2)}_s - V_s = 0 &- \int_s^T \left(Z^{(2)}_r - \tilde{Z}_r\right) \mathrm{d}W_r - \int_s^T \left(-2Z^{(0)}_r\right)\left(Y^{(2)}_r Z^{(2)}_r - V_r \tilde{Z}_r\right) \mathrm{d}r = \\ &= 0 - \int_s^T \left(Z^{(2)}_r - \tilde{Z}_r\right) \mathrm{d}W_r - \int_s^T \left(-2Z^{(0)}_r\right)\left(\left(Y^{(2)}_r - V_r\right) Z^{(2)}_r + V_r\left(Z^{(2)}_r - \tilde{Z}_r\right)\right) \mathrm{d}r. \end{aligned}$$

Using the boundedness of $Z^{(0)}$, $Z^{(2)}$ and V this implies using Lemma A.1.7 that $Y^{(2)} - V$ is 0 almost everywhere. Therefore, after setting $\tilde{W}_s := W_s - \int_t^s 2Z^{(0)}_r V_r \,\mathrm{d}r$, $s \in [t, T]$ we get from the above equation $\int_s^T \left(Z^{(2)}_r - \tilde{Z}_r\right) \mathrm{d}\tilde{W}_r = 0$ a.s. for $s \in [t, T]$. Since $\tilde{W}$ is a Brownian motion under some probability measure equivalent to $\mathbb{P}$ we also have $Z^{(2)} - \tilde{Z} = 0$ a.e.
Similarly one shows that $Y^{(1)}$ and U as well as $Z^{(1)}$ and $\check{Z}$ coincide so

$$Y^{(1)} = U, \quad Y^{(2)} = V, \quad Z^{(1)} = \check{Z} \quad \text{and} \quad Z^{(2)} = \tilde{Z} \qquad \text{a.e.}$$

Now, remember $U_s = \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u(s, X^{(1)}_s, X^{(2)}_s)$. Together with $u^{(1)}(s, X^{(1)}_s, X^{(2)}_s) = Y^{(1)}_s$ and $Y^{(1)} = U$ this yields $u^{(1)}(t, \cdot) = \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u(t, \cdot)$ and, therefore, $u^{(1)} = \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u$ a.e. on $I^M_{\max}$. Similarly, we get $u^{(2)} = \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} u$. ✓

Now, note that $u^{(1)} = \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u$ is continuous. This makes Lemma 4.3.5 applicable, so

$$Z^{(0)} = Z = U = Y^{(1)} \text{ a.e.} \tag{4.14}$$

Thereby $Y^{(1)}$, $Y^{(2)}$ satisfy the following dynamics:

$$Y_s^{(1)} = g'(X_T^{(1)}) - \int_s^T Z_r^{(1)} \,\mathrm{d}W_r - \int_s^T \left(-2Y_r^{(1)} Y_r^{(2)}\right) Z_r^{(1)} \,\mathrm{d}r, \tag{4.15}$$

$$Y_s^{(2)} = -\delta'(X_T^{(2)}) - \int_s^T Z_r^{(2)} \,\mathrm{d}W_r - \int_s^T \left(-2Y_r^{(1)} Y_r^{(2)}\right) Z_r^{(2)} \,\mathrm{d}r, \quad s \in [t, T], \tag{4.16}$$

which implies using the chain rule of Lemma A.3.1:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x^{(i)}} Y_s^{(1)} =& g''(X_T^{(1)}) \frac{\mathrm{d}}{\mathrm{d}x^{(i)}} X_T^{(1)} - \int_s^T \frac{\mathrm{d}}{\mathrm{d}x^{(i)}} Z_r^{(1)} \,\mathrm{d}W_r \\ &- \int_s^T (-2) \left(\left(\frac{\mathrm{d}}{\mathrm{d}x^{(i)}} Y_r^{(1)} Y_r^{(2)} + Y_r^{(1)} \frac{\mathrm{d}}{\mathrm{d}x^{(i)}} Y_r^{(2)} \right) Z_r^{(1)} + Y_r^{(1)} Y_r^{(2)} \frac{\mathrm{d}}{\mathrm{d}x^{(i)}} Z_r^{(1)} \right) \mathrm{d}r, \end{aligned}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x^{(i)}} Y_s^{(2)} =& -\delta''(X_T^{(2)}) \frac{\mathrm{d}}{\mathrm{d}x^{(i)}} X_T^{(2)} - \int_s^T \frac{\mathrm{d}}{\mathrm{d}x^{(i)}} Z_r^{(2)} \,\mathrm{d}W_r \\ &- \int_s^T (-2) \left(\left(\frac{\mathrm{d}}{\mathrm{d}x^{(i)}} Y_r^{(1)} Y_r^{(2)} + Y_r^{(1)} \frac{\mathrm{d}}{\mathrm{d}x^{(i)}} Y_r^{(2)} \right) Z_r^{(2)} + Y_r^{(1)} Y_r^{(2)} \frac{\mathrm{d}}{\mathrm{d}x^{(i)}} Z_r^{(2)} \right) \mathrm{d}r, \end{aligned}$$

for $i = 1, 2$. Let us recall some statements about the forward process obtained in the proof of Lemma 4.3.1:

$$\frac{\mathrm{d}}{\mathrm{d}x^{(2)}} X^{(2)} > 0, \quad \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} X^{(1)} = 1, \quad \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} X^{(1)} = 0, \quad \text{a.e.},$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}} X_s^{(2)} \right)^{-1} = 1 - \int_t^s 2Y_r^{(1)} Z_r^{(2)} \left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}} X_r^{(2)} \right)^{-1} \mathrm{d}r, \tag{4.17}$$

$$\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} X_s^{(2)} = \int_t^s 2Y_r^{(1)} \left(Z_r^{(1)} + Z_r^{(2)} \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} X_r^{(2)} \right) \mathrm{d}r. \tag{4.18}$$

Using the chain rule of Lemma A.3.1 and the decoupling condition, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} Y_s^{(i)} &= \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u^{(i)}(s, X_s^{(1)}, X_s^{(2)}) + \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} u^{(i)}(s, X_s^{(1)}, X_s^{(2)}) \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} X_s^{(2)}, \\ \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} Y_s^{(i)} &= \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} u^{(i)}(s, X_s^{(1)}, X_s^{(2)}) \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} X_s^{(2)}, \quad i = 1, 2. \end{aligned}$$

Now, define

$$Y_s^{(12)} := \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} u^{(1)}(s, X_s^{(1)}, X_s^{(2)}) = \left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}} Y_s^{(1)} \right) \left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}} X_s^{(2)} \right)^{-1}, \tag{4.19}$$

$$Y_s^{(22)} := \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} u^{(2)}(s, X_s^{(1)}, X_s^{(2)}) = \left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}} Y_s^{(2)} \right) \left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}} X_s^{(2)} \right)^{-1},$$

$$Y_s^{(11)} := \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u^{(1)}(s, X_s^{(1)}, X_s^{(2)}) = \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} Y_s^{(1)} - Y_s^{(12)} \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} X_s^{(2)}, \tag{4.20}$$

$$Y_s^{(21)} := \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u^{(2)}(s, X_s^{(1)}, X_s^{(2)}) = \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} Y_s^{(2)} - Y_s^{(22)} \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} X_s^{(2)}.$$

We can apply the Itô formula to deduce dynamics of $Y^{(12)}$ and $Y^{(11)}$ from dynamics of $\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}Y^{(1)}$, $\left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X^{(2)}\right)^{-1}$, $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}Y^{(1)}$ and $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X^{(2)}$:
Let us define $Z_s^{(12)} := \left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}Z_s^{(1)}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_s^{(2)}\right)^{-1}$, so we can use (4.19) to write

$$\begin{aligned}Y_s^{(12)} =& 0 - \int_s^T Z_r^{(12)}\,\mathrm{d}W_r - \int_s^T \Bigg\{(-2)\left(\left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}Y_r^{(1)}Y_r^{(2)} + Y_r^{(1)}\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}Y_r^{(2)}\right)Z_r^{(1)}\right.\\ &\left. + Y_r^{(1)}Y_r^{(2)}\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}Z_r^{(1)}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_r^{(2)}\right)^{-1} - 2\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}Y_s^{(1)}Y_r^{(1)}Z_r^{(2)}\left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_r^{(2)}\right)^{-1}\Bigg\}\,\mathrm{d}r.\end{aligned}$$

Using the definitions of $Y^{(12)}$, $Y^{(22)}$, $Z^{(12)}$ we can simplify this to

$$\begin{aligned}Y_s^{(12)} =& 0 - \int_s^T Z_r^{(12)}\,\mathrm{d}W_r\\ &- \int_s^T (-2)\left(\left(Y_r^{(12)}Y_r^{(2)} + Y_r^{(1)}Y_r^{(22)}\right)Z_r^{(1)} + Y_r^{(1)}Y_r^{(2)}Z_r^{(12)} + Y_r^{(12)}Y_r^{(1)}Z_r^{(2)}\right)\mathrm{d}r.\end{aligned}$$

Let us now define $Z_s^{(11)} := \frac{\mathrm{d}}{\mathrm{d}x^{(1)}}Z_s^{(1)} - Z_s^{(12)}\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_s^{(2)}$, so we can write using (4.20)

$$\begin{aligned}Y_s^{(11)} =& g''(X_T^{(1)}) - \int_s^T Z_r^{(11)}\,\mathrm{d}W_r\\ &- \int_s^T \Bigg\{(-2)\left(\left(\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}Y_r^{(1)}\,Y_r^{(2)} + Y_r^{(1)}\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}Y_r^{(2)}\right)Z_r^{(1)} + Y_r^{(1)}Y_r^{(2)}\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}Z_r^{(1)}\right)\\ &- (-2)\left(\left(Y_r^{(12)}\,Y_r^{(2)} + Y_r^{(1)}Y_r^{(22)}\right)Z_r^{(1)} + Y_r^{(1)}Y_r^{(2)}Z_r^{(12)} + Y_r^{(12)}Y_r^{(1)}Z_r^{(2)}\right)\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_r^{(2)}\\ &- Y_r^{(12)}\cdot 2\cdot Y_r^{(1)}\left(Z_r^{(1)} + Z_r^{(2)}\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_r^{(2)}\right)\Bigg\}\,\mathrm{d}r.\end{aligned}$$

The two marked terms above can be effectively merged into one using (4.20):

$$\begin{aligned}Y_s^{(11)} =& g''(X_T^{(1)}) - \int_s^T Z_r^{(11)}\,\mathrm{d}W_r\\ &- \int_s^T \Bigg\{(-2)\left(\left(Y_r^{(11)}Y_r^{(2)} + Y_r^{(1)}\,\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}Y_r^{(2)}\right)Z_r^{(1)} + Y_r^{(1)}Y_r^{(2)}\,\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}Z_r^{(1)}\right)\\ &- (-2)\left(Y_r^{(1)}\,Y_r^{(22)}\,Z_r^{(1)} + Y_r^{(1)}Y_r^{(2)}\,Z_r^{(12)} + Y_r^{(12)}Y_r^{(1)}Z_r^{(2)}\right)\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_r^{(2)}\\ &- Y_r^{(12)}\cdot 2\cdot Y_r^{(1)}\left(Z_r^{(1)} + Z_r^{(2)}\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_r^{(2)}\right)\Bigg\}\,\mathrm{d}r.\end{aligned}$$

Similarly, the four marked terms can be merged into only two using the structure of $Y^{(21)}$ and $Z^{(11)}$ s.t.

$$\begin{aligned}Y_s^{(11)} =& g''(X_T^{(1)}) - \int_s^T Z_r^{(11)}\,\mathrm{d}W_r\\ &- \int_s^T \Bigg\{(-2)\left(\left(Y_r^{(11)}Y_r^{(2)} + Y_r^{(1)}Y_r^{(21)}\right)Z_r^{(1)} + Y_r^{(1)}Y_r^{(2)}Z_r^{(11)}\right)\\ &- (-2)\left(Y_r^{(12)}Y_r^{(1)}\,Z_r^{(2)}\right)\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_r^{(2)} - Y_r^{(12)}\cdot 2\cdot Y_r^{(1)}\left(Z_r^{(1)} + \,Z_r^{(2)}\,\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_r^{(2)}\right)\Bigg\}\,\mathrm{d}r,\end{aligned}$$

where the two marked terms effectively cancel each other out:

$$
\begin{aligned}
Y_s^{(11)} =& g''(X_T^{(1)}) - \int_s^T Z_r^{(11)}\,\mathrm{d}W_r \\
&- \int_s^T (-2)\left(\left(Y_r^{(11)}Y_r^{(2)} + Y_r^{(1)}Y_r^{(21)}\right) Z_r^{(1)} + Y_r^{(1)}Y_r^{(2)}Z_r^{(11)} + Y_r^{(12)}Y_r^{(1)}Z_r^{(1)}\right)\,\mathrm{d}r.
\end{aligned}
$$

Analogously to $Y^{(12)}$ we can deduce dynamics of $Y^{(22)}$:

$$
\begin{aligned}
Y_s^{(22)} =& -\delta''(X_T^{(2)}) - \int_s^T Z_r^{(22)}\,\mathrm{d}W_r \\
&- \int_s^T (-2)\left(\left(Y_r^{(12)}Y_r^{(2)} + Y_r^{(1)}Y_r^{(22)}\right) Z_r^{(2)} + Y_r^{(1)}Y_r^{(2)}Z_r^{(22)} + Y_r^{(22)}Y_r^{(1)}Z_r^{(2)}\right)\,\mathrm{d}r.
\end{aligned}
$$

From here we can, analogously to $Y^{(11)}$, deduce dynamics of $Y^{(21)}$:

$$
\begin{aligned}
Y_s^{(21)} =& 0 - \int_s^T Z_r^{(21)}\,\mathrm{d}W_r \\
&- \int_s^T (-2)\left(\left(Y_r^{(11)}Y_r^{(2)} + Y_r^{(1)}Y_r^{(21)}\right) Z_r^{(2)} + Y_r^{(1)}Y_r^{(2)}Z_r^{(21)} + Y_r^{(22)}Y_r^{(1)}Z_r^{(1)}\right)\,\mathrm{d}r.
\end{aligned}
$$

And so we have finally obtained the complete dynamics of the 4-dimensional process $(Y^{(ij)})$, $i,j=1,2$, which are clearly linear in it. Furthermore, remember:

- $Y^{(1)}$, $Y^{(2)}$ are uniformly bounded independently of (t,x) due to the decoupling condition, $u^{(i)} = \frac{\mathrm{d}}{\mathrm{d}x^{(i)}}u$, $i=1,2$ and Lemma 4.3.1,
- $Z^{(1)}$, $Z^{(2)}$ are $BMO(\mathbb{P})$ processes with uniformly bounded $BMO(\mathbb{P})$-norms independently of (t,x) due to (4.15), (4.16) and Theorem A.1.11,
- $(Y^{(ij)})$, $i,j=1,2$ are bounded according to their definition (with a bound which may depend on t,x at this point),
- $(Z^{(ij)})$, $i,j=1,2$ are in $BMO(\mathbb{P})$ according to Theorem A.1.11,
- $(Y_T^{(ij)})_{i,j=1,2}$ is uniformly bounded by $\|g''\|_\infty + \|\delta''\|_\infty < \infty$.

Therefore, Lemma A.1.7 is applicable and $(Y^{(ij)})_{i,j=1,2}$ is uniformly bounded, independently of (t,x). In particular, $Y_t^{(ij)} = \frac{\mathrm{d}}{\mathrm{d}x^{(j)}}u^{(i)}(t,x)$, $i,j=1,2$ can be controlled independently of $t \in I_{\max}^M$, $x \in \mathbb{R}^2$, while $\frac{\mathrm{d}}{\mathrm{d}x^{(j)}}u^{(0)}(t,x)$, $j=1,2$ has the same property as we already know. This shows $I_{\max}^M = [0,T]$ using Lemma 4.2.29. $\square$

Lemma 4.3.8. *Assume that g, δ, g', δ' are Lipschitz continuous. Let $\left(u^{(i)}\right)_{i=0,1,2}$ be the unique weakly regular Markovian decoupling field to the problem (4.13) constructed in Theorem 4.3.7.*
Assume that $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(i)}$, $i=0,1,2$, has a version whose restriction to the set $[t,T)\times\mathbb{R}^2$ is continuous in the first two components $(s,x^{(1)})$ for some $t\in[0,T)$. Then for any initial condition $(X_t^{(1)},X_t^{(2)})^\top = (x^{(1)},x^{(2)})^\top = x \in \mathbb{R}^2$ the associated processes $Z^{(i)}$, $i=0,1,2$, on $[t,T]$ satisfy

$$
Z_s^{(i)}(\omega) = \frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(i)}\left(s, X_s^{(1)}(\omega), X_s^{(2)}(\omega)\right), \quad i=0,1,2,
$$

for almost all $(s,\omega) \in [t,T] \times \Omega$.
Furthermore, in this case the associated processes

$$\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} X^{(2)}, \quad \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} X^{(2)} \quad \textit{and} \quad \left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}} X^{(2)}\right)^{-1} \quad \textit{on } [t,T],$$

can be bounded uniformly, i.e. independently of t, x.

Proof. The first part of the proof works analogously to the proof of Lemma 4.3.5 and thus we keep our argumentation short. For $i = 0,1,2$ we consider

$$\frac{1}{h}\mathbb{E}[Y^{(i)}_{s+h}(W_{s+h} - W_s)|\mathcal{F}_s]$$

for small $h > 0$. As in proof of Lemma 4.3.5 we use Itô's formula applied to (4.13) to obtain for $i = 1,2$

$$\begin{aligned} Y^{(i)}_{s+h}(W_{s+h} - W_s) = &\int_s^{s+h} Y^{(i)}_r \,\mathrm{d}W_r + \int_s^{s+h} (W_r - W_s) Z^{(i)}_r \,\mathrm{d}W_r + \\ &+ \int_s^{s+h} (W_r - W_s)\left(-2Z^{(0)}_r Y^{(2)}_r\right) Z^{(i)}_r \,\mathrm{d}r + \int_s^{s+h} Z^{(i)}_r \,\mathrm{d}r, \end{aligned}$$

and also

$$Y^{(0)}_{s+h}(W_{s+h} - W_s) = \int_s^{s+h} Y^{(0)}_r \,\mathrm{d}W_r + \int_s^{s+h} (W_r - W_s) Z^{(0)}_r \,\mathrm{d}W_r + \int_s^{s+h} Z^{(0)}_r \,\mathrm{d}r,$$

which leads to

$$\frac{1}{h}\mathbb{E}[Y^{(0)}_{s+h}(W_{s+h} - W_s)|\mathcal{F}_s] = \frac{1}{h}\mathbb{E}\left[\int_s^{s+h} Z^{(0)}_r \,\mathrm{d}r \middle| \mathcal{F}_s\right] \to Z^{(0)}_s \quad \text{for} \quad h \to 0.$$

and

$$\frac{1}{h}\mathbb{E}[Y^{(i)}_{s+h}(W_{s+h} - W_s)|\mathcal{F}_s] = \frac{1}{h}\mathbb{E}\left[\int_s^{s+h} Z^{(i)}_r \left(1 + (W_r - W_s)\left(-2Z^{(0)}_r Y^{(2)}_r\right)\right) \mathrm{d}r \middle| \mathcal{F}_s\right] \to Z^{(i)}_s$$

as $h \to 0$ for $i = 1,2$. The argumentation works for almost all $s \in [t,T]$.

On the other hand we can use the decoupling condition to rewrite

$$\begin{aligned} Y^{(i)}_{s+h}(W_{s+h} - W_s) =& u^{(i)}\left(s+h, X^{(1)}_{s+h}, X^{(2)}_s\right)(W_{s+h} - W_s) \\ &+ \left(u^{(i)}\left(s+h, X^{(1)}_{s+h}, X^{(2)}_{s+h}\right) - u^{(i)}\left(s+h, X^{(1)}_{s+h}, X^{(2)}_s\right)\right)(W_{s+h} - W_s), \end{aligned}$$

for $i = 0,1,2$. Let us deal separately with the two summands. For the first summand recall that

- $X^{(1)}_s, X^{(2)}_s$ are $\mathcal{F}_s$ measurable,
- $X^{(1)}_{s+h} = X^{(1)}_s + (W_{s+h} - W_s)$,
- $W_{s+h} - W_s$ is independent of $\mathcal{F}_s$,
- u is deterministic, i.e. is assumed to be a function of $\left(s, x^{(1)}, x^{(2)}\right) \in [0,T] \times \mathbb{R}^2$.

Combining these properties leads to

$$\lim_{h\downarrow 0}\frac{1}{h}\mathbb{E}\left[u^{(i)}\left(s+h,X_{s+h}^{(1)},X_s^{(2)}\right)(W_{s+h}-W_s)\Big|\mathcal{F}_s\right]=\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(i)}\left(s,X_s^{(1)},X_s^{(2)}\right),$$

if $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(i)}$ is continuous in the first two components on $[t,T)\times\mathbb{R}^2$, where we use that $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(i)}$ is bounded.
For the second summand remember that

- $u^{(i)}$ is also Lipschitz continuous in the last component with some Lipschitz constant L,
- $X_{s+h}^{(2)}=X_s^{(2)}+\int_s^{s+h}\left(Z_r^{(0)}\right)^2\mathrm{d}r$.

These properties allow us to estimate

$$\begin{aligned}&\frac{1}{h}\left|\mathbb{E}\left[\left(u^{(i)}\left(s+h,X_{s+h}^{(1)},X_{s+h}^{(2)}\right)-u^{(i)}\left(s+h,X_{s+h}^{(1)},X_s^{(2)}\right)\right)(W_{s+h}-W_s)\Big|\mathcal{F}_s\right]\right|\\&\le\frac{1}{h}\mathbb{E}\left[L\cdot\left(\int_s^{s+h}\left(Z_r^{(0)}\right)^2\mathrm{d}r\right)\cdot|W_{s+h}-W_s|\Big|\mathcal{F}_s\right]\le\frac{1}{h}L\cdot h\|Z^{(0)}\|_\infty^2\mathbb{E}[|W_{s+h}-W_s|],\end{aligned}$$

which tends to 0 as $h\to 0$.
Therefore, we can conclude

$$Z_s^{(i)}=\lim_{h\downarrow 0}\frac{1}{h}\mathbb{E}[Y_{s+h}^{(i)}(W_{s+h}-W_s)|\mathcal{F}_s]=\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(i)}\left(s,X_s^{(1)},X_s^{(2)}\right)$$

if $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(i)}$ is continuous in the first two components on $[t,T)\times\mathbb{R}^2$, for $i=0,1,2$.

Now, remember (4.17) and (4.18) from the proof of Theorem 4.3.7:

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_s^{(2)}\right)^{-1}&=1-\int_t^s 2Y_r^{(1)}Z_r^{(2)}\left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_r^{(2)}\right)^{-1}\mathrm{d}r,\\\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_s^{(2)}&=\int_t^s 2Y_r^{(1)}\left(Z_r^{(1)}+Z_r^{(2)}\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_r^{(2)}\right)\mathrm{d}r,\end{aligned}$$

a.s. for $s\in[t,T]$. The first equation implies

$$\left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_s^{(2)}\right)^{-1}=\exp\left(-\int_t^s 2Y_r^{(1)}Z_r^{(2)}\,\mathrm{d}r\right),$$

which using $Z^{(2)}=\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(2)}(\cdot,X_\cdot^{(1)},X_\cdot^{(2)})$, $Y^{(1)}=Z^{(0)}=\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(0)}(\cdot,X_\cdot^{(1)},X_\cdot^{(2)})$ (see (4.14) in the proof of Theorem 4.3.7) and uniform boundedness of $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(i)}$ for $i=0,1,2$ implies uniform boundedness of $\left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_s^{(2)}\right)^{-1}$ and its inverse $\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}X_s^{(2)}$.
Furthermore,

$$\left|\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_s^{(2)}\right|\le 2T\|Y^{(1)}Z^{(1)}\|_\infty+\int_t^s 2\|Y^{(1)}Z^{(2)}\|_\infty\left|\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X_r^{(2)}\right|\mathrm{d}r,$$

which using Gronwall's lemma together with uniform boundedness of $Z^{(2)}$, $Y^{(1)}$ and

$$Z^{(1)}=\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(1)}(\cdot,X_\cdot^{(1)},X_\cdot^{(2)})$$

implies uniform boundedness of $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}X^{(2)}$.

□

For the coming result we employ the following notation:

- For a real number $H>0$ let $\chi_H:\mathbb{R}\to\mathbb{R}$ be defined via $\chi_H(x):=(-H)\vee(x\wedge H)$ for $x\in\mathbb{R}$. In particular, χ_H is bounded, Lipschitz continuous with 1 as Lipschitz constant and coincides with the identity function on the interval $[-H,H]$.
- For real numbers $y^{(ij)}$, $i,j=1,2$ and $y^{(i)}$, $i=1,2$ we denote by $y^{(ij)\wedge H}$ and $y^{(i)\wedge H}$ the values $\chi_H(y^{(ij)})$ and $\chi_H(y^{(i)})$.

Now, assume that g, δ, g', δ', g'', δ'' are all Lipschitz continuous and consider for $H>0$ the following MLLC system with $d=1$, $n=2$ and $m=6$:

$$
\begin{aligned}
X_s^{(1)} &= x^{(1)}+\int_t^s 1\,\mathrm{d}W_r,\\
X_s^{(2)} &= x^{(2)}+\int_t^s \left(Z_r^{(0)}\right)^2\mathrm{d}r,\\
Y_s^{(0)} &= g(X_T^{(1)})-\delta(X_T^{(2)})-\int_s^T Z_r^{(0)}\,\mathrm{d}W_r, && u^{(0)}(s,X_s^{(1)},X_s^{(2)})=Y_s^{(0)},\\
Y_s^{(1)} &= g'(X_T^{(1)})-\int_s^T Z_r^{(1)}\,\mathrm{d}W_r-\int_s^T\left(-2Z_r^{(0)}Y_r^{(2)}\right)Z_r^{(1)}\,\mathrm{d}r, && u^{(1)}(s,X_s^{(1)},X_s^{(2)})=Y_s^{(1)},\\
Y_s^{(2)} &= -\delta'(X_T^{(2)})-\int_s^T Z_r^{(2)}\,\mathrm{d}W_r-\int_s^T\left(-2Z_r^{(0)}Y_r^{(2)}\right)Z_r^{(2)}\,\mathrm{d}r, && u^{(2)}(s,X_s^{(1)},X_s^{(2)})=Y_s^{(2)},
\end{aligned}
$$

and

$$
\begin{aligned}
Y_s^{(11)} =& g''(X_T^{(1)})-\int_s^T Z_r^{(11)}\,\mathrm{d}W_r-\int_s^T(-2)\Big\{\left(Y_r^{(11)\wedge H}Y_r^{(2)\wedge H}+Y_r^{(1)\wedge H}Y_r^{(21)\wedge H}\right)Z_r^{(1)}\\
&+Y_r^{(1)\wedge H}Y_r^{(2)\wedge H}Z_r^{(11)}+Y_r^{(12)\wedge H}Y_r^{(1)\wedge H}Z_r^{(1)}\Big\}\,\mathrm{d}r,\\
Y_s^{(12)} =& 0-\int_s^T Z_r^{(12)}\,\mathrm{d}W_r-\int_s^T(-2)\Big\{\left(Y_r^{(12)\wedge H}Y_r^{(2)\wedge H}+Y_r^{(1)\wedge H}Y_r^{(22)\wedge H}\right)Z_r^{(1)}\\
&+Y_r^{(1)\wedge H}Y_r^{(2)\wedge H}Z_r^{(12)}+Y_r^{(12)\wedge H}Y_r^{(1)\wedge H}Z_r^{(2)}\Big\}\,\mathrm{d}r,\\
Y_s^{(21)} =& 0-\int_s^T Z_r^{(21)}\,\mathrm{d}W_r-\int_s^T(-2)\Big\{\left(Y_r^{(11)\wedge H}Y_r^{(2)\wedge H}+Y_r^{(1)\wedge H}Y_r^{(21)\wedge H}\right)Z_r^{(2)}\\
&+Y_r^{(1)\wedge H}Y_r^{(2)\wedge H}Z_r^{(21)}+Y_r^{(22)\wedge H}Y_r^{(1)\wedge H}Z_r^{(1)}\Big\}\,\mathrm{d}r,\\
Y_s^{(22)} =& -\delta''(X_T^{(2)})-\int_s^T Z_r^{(22)}\,\mathrm{d}W_r-\int_s^T(-2)\Big\{\left(Y_r^{(12)\wedge H}Y_r^{(2)\wedge H}+Y_r^{(1)\wedge H}Y_r^{(22)\wedge H}\right)Z_r^{(2)}\\
&+Y_r^{(1)\wedge H}Y_r^{(2)\wedge H}Z_r^{(22)}+Y_r^{(22)\wedge H}Y_r^{(1)\wedge H}Z_r^{(2)}\Big\}\,\mathrm{d}r,
\end{aligned}
$$

with the decoupling conditions

$$
\begin{aligned}
&u^{(11)}(s,X_s^{(1)},X_s^{(2)})=Y_s^{(11)}, && u^{(12)}(s,X_s^{(1)},X_s^{(2)})=Y_s^{(12)},\\
&u^{(21)}(s,X_s^{(1)},X_s^{(2)})=Y_s^{(21)}, && u^{(22)}(s,X_s^{(1)},X_s^{(2)})=Y_s^{(22)}.
\end{aligned}
\tag{4.21}
$$

With (4.21) we will always refer to all the above equations belonging to the current MLLC system.

Theorem 4.3.9. *For sufficiently large $H > 0$ the above problem (4.21) satisfies $I_{\max}^M = [0, T]$ and in addition*

$$u^{(0)} = u, \quad u^{(1)} = \frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u, \quad u^{(2)} = \frac{\mathrm{d}}{\mathrm{d}x^{(2)}}u, \quad u^{(11)} = \frac{\mathrm{d}^2}{\left(\mathrm{d}x^{(1)}\right)^2}u,$$

$$u^{(12)} = \frac{\mathrm{d}}{\mathrm{d}x^{(2)}}\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u, \quad u^{(21)} = \frac{\mathrm{d}}{\mathrm{d}x^{(1)}}\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}u, \quad u^{(22)} = \frac{\mathrm{d}^2}{\left(\mathrm{d}x^{(2)}\right)^2}u,$$

a.e. where u is the unique weakly regular Markovian decoupling field to the problem (4.8).
In particular, u is three times weakly differentiable w.r.t. x with uniformly bounded derivatives.

Proof. The proof is in parts akin to the proof of Lemma 4.3.1 and we will seek to keep these parts short.
Assume $I_{\max}^M = (t_{\min}^M, T]$. Let $u^{(i)}$, $u^{(jk)}$, $i = 0, 1, 2$, $j, k = 1, 2$ be the associated weakly regular Markovian decoupling field on $I_{\max}^M$. Let $t \in I_{\max}^M$. We want to control $\frac{\mathrm{d}}{\mathrm{d}x}u^{(i)}u(t, \cdot)$, $\frac{\mathrm{d}}{\mathrm{d}x}u^{(jk)}(t, \cdot)$, $i = 0, 1, 2$, $j, k = 1, 2$ independently of t creating a contradiction according to Lemma 4.2.29.

Now, consider the first three components of the decoupling field: Since these components $u^{(i)}$, $i = 0, 1, 2$ clearly constitute a weakly regular Markovian decoupling field to the problem (4.13)

- the mapping $\left(u^{(i)}\right)_{i=0,1,2}$ in (4.13) and in (4.21) is the same function on $[t, T]$ according to Theorem 4.2.24 (for every $H > 0$),
- the processes $X^{(1)}$, $X^{(2)}$, $Y^{(i)}$, $Z^{(i)}$, $i = 0, 1, 2$ in (4.13) must coincide with the same-named processes in (4.21) according to strong regularity. This is true for every $t \in I_{\max}^M$ and initial condition $x \in \mathbb{R}^2$.

So, we can apply Theorem 4.3.7 and get

$$u^{(0)} = u, \quad u^{(1)} = \frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u, \quad u^{(2)} = \frac{\mathrm{d}}{\mathrm{d}x^{(2)}}u \quad \text{on} \quad I_{\max}^M,$$

where $I_{\max}^M$ is the maximal interval for the problem given by (4.21). In particular, the last two of the three functions above are uniformly bounded.
Furthermore, we saw in the proof of Theorem 4.3.7 that

- $Y^{(1)}$, $Y^{(2)}$ are uniformly bounded independently of (t, x),
- $Z^{(1)}$, $Z^{(2)}$ are $BMO(\mathbb{P})$ processes with uniformly bounded $BMO(\mathbb{P})$-norms independently of (t, x).

In particular, $Y^{(i)\wedge H} = Y^{(i)}$ for $i = 1, 2$ if we make H large enough. We will make this assumption from now on.
The processes $Y^{(jk)}$, $j, k = 1, 2$ satisfy

$$Y_s^{(jk)} = Y_T^{(jk)} - \int_s^T Z_r^{(jk)}\,\mathrm{d}W_r - \\ - \int_s^T \left(\sum_{l_1,l_2,l_3,l_4=1,2} \alpha_{l_1,l_2,l_3,l_4}^{(jk)} Y_r^{(l_1)} Z_r^{(l_2)} Y_r^{(l_3 l_4)\wedge H} - 2Y_r^{(1)} Y_r^{(2)} Z_r^{(jk)} \right) \mathrm{d}r,$$

where $\alpha_{l_1,l_2,l_3,l_4}^{(jk)}$ is always either 0 or -2. Since, due to the structure of the terminal condition, $Y_T^{(jk)}$ are uniformly bounded we can apply Lemma A.1.7 to obtain uniform boundedness of $Y^{(jk)}$ as processes on $[t, T]$ independently of t, x: To apply Lemma A.1.7 we use that

- $Z^{(jk)}$, $j,k=1,2$ are bounded, since we work with Markovian decoupling fields,
- $Y^{(jk)}$, $j,k=1,2$ are bounded due to their dynamics, the fact that $Z^{(l_2)}$ is always in $BMO(\mathbb{P})$ and Lemma A.1.10.

In particular, $Y^{(jk)\wedge H} = Y^{(jk)}$ for $j,k=1,2$ if we make H large enough. We will make this assumption from now on.
This implies that the processes $Y^{(jk)}$, $j,k=1,2$ must coincide with the same-named processes in the proof of Theorem 4.3.7 since

- they satisfy the same dynamics with $Y^{(1)}, Y^{(2)}, Z^{(1)}, Z^{(2)}$ being the same as already mentioned,
- they satisfy the same terminal condition and
- we can apply Lemma A.1.7 to the difference of these four-dimensional processes obtaining that this difference must vanish.

This implies, however, that $Y_t^{(jk)} = \frac{\mathrm{d}}{\mathrm{d}x^{(k)}}u^{(j)}\left(t,x^{(1)},x^{(2)}\right)$ for almost all $x^{(1)},x^{(2)}$. So, we obtain $u^{(jk)} = \frac{\mathrm{d}}{\mathrm{d}x^{(k)}}u^{(j)}$, $j,k=1,2$ a.e. In particular, these functions are uniformly bounded according to Theorem 4.3.7.
According to Remark 4.2.21 the functions $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u = u^{(1)}$, $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(i)} = u^{(i1)}$, $i=1,2$ are continuous on $[t,T]\times\mathbb{R}^2$ and we can apply Lemma 4.3.8:

$$Z^{(i)} = \frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(i)}\left(\cdot, X_\cdot^{(1)}, X_\cdot^{(2)}\right), \quad i=0,1,2.$$

So, $Z^{(i)}$, $i=0,1,2$ are uniformly bounded.
Let us now analyze higher order derivatives $\frac{\mathrm{d}}{\mathrm{d}x^{(i)}}u^{(jk)}$, $i,j,k=1,2$. As usual this is done by investigating dynamics of $\frac{\mathrm{d}}{\mathrm{d}x^{(i)}}Y^{(jk)}$, $i,j,k=1,2$. Using strong regularity

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x^{(i)}}Y_s^{(jk)} = \frac{\mathrm{d}}{\mathrm{d}x^{(i)}}Y_T^{(jk)} - \int_s^T \frac{\mathrm{d}}{\mathrm{d}x^{(i)}}Z_r^{(jk)}\,\mathrm{d}W_r - \\ - \int_s^T \left(-2G_r^{(jk)} + \sum_{l_1,l_2,l_3,l_4=1,2} \alpha^{(jk)}_{l_1,l_2,l_3,l_4} H_r^{(jk),l_1,l_2,l_3,l_4}\right)\mathrm{d}r,\end{aligned}$$

where

$$H_r^{i,(jk),l_1,l_2,l_3,l_4} = \frac{\mathrm{d}}{\mathrm{d}x^{(i)}}Y_r^{(l_1)}Z_r^{(l_2)}Y_r^{(l_3l_4)} + Y_r^{(l_1)}\frac{\mathrm{d}}{\mathrm{d}x^{(i)}}Z_r^{(l_2)}Y_r^{(l_3l_4)} + Y_r^{(l_1)}Z_r^{(l_2)}\frac{\mathrm{d}}{\mathrm{d}x^{(i)}}Y_r^{(l_3l_4)},$$

$$G_r^{i,(jk)} = \frac{\mathrm{d}}{\mathrm{d}x^{(i)}}Y_r^{(1)}Y_r^{(2)}Z_r^{(jk)} + Y_r^{(1)}\frac{\mathrm{d}}{\mathrm{d}x^{(i)}}Y_r^{(2)}Z_r^{(jk)} + Y_r^{(1)}Y_r^{(2)}\frac{\mathrm{d}}{\mathrm{d}x^{(i)}}Z_r^{(jk)}.$$

This already implies that $\frac{\mathrm{d}}{\mathrm{d}x^{(i)}}Y^{(jk)}$, $i,j,k=1,2$ is uniformly bounded according to Lemma A.1.7, which is applicable since:

- $\frac{\mathrm{d}}{\mathrm{d}x^{(i)}}Y_T^{(jk)}$ is either 0 or has the structure $g^{(3)}(X_T^{(1)})\frac{\mathrm{d}}{\mathrm{d}x^{(i)}}X_T^{(1)}$ or $-\delta^{(3)}(X_T^{(2)})\frac{\mathrm{d}}{\mathrm{d}x^{(i)}}X_T^{(2)}$, which is always uniformly bounded according to the Lipschitz continuity of g'', δ'' and Lemma 4.3.8,
- $\frac{\mathrm{d}}{\mathrm{d}x^{(i)}}Y_r^{(l)} = \frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(l)}(r,X_r^{(1)},X_r^{(2)})\frac{\mathrm{d}}{\mathrm{d}x^{(i)}}X_r^{(1)} + \frac{\mathrm{d}}{\mathrm{d}x^{(2)}}u^{(l)}(r,X_r^{(1)},X_r^{(2)})\frac{\mathrm{d}}{\mathrm{d}x^{(i)}}X_r^{(2)}$ is also uniformly bounded according to Theorem 4.3.7 and Lemma 4.3.8,
- $\frac{\mathrm{d}}{\mathrm{d}x^{(i)}}Y_r^{(jk)} = \frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(jk)}(r,X_r^{(1)},X_r^{(2)})\frac{\mathrm{d}}{\mathrm{d}x^{(i)}}X_r^{(1)} + \frac{\mathrm{d}}{\mathrm{d}x^{(2)}}u^{(jk)}(r,X_r^{(1)},X_r^{(2)})\frac{\mathrm{d}}{\mathrm{d}x^{(i)}}X_r^{(2)}$ is a bounded processes on $[t,T]$ according to Lemma 4.3.8 (but not necessarily uniformly in t at this point),

- $\frac{\mathrm{d}}{\mathrm{d}x^{(i)}} Z_r^{(l)} = \frac{\mathrm{d}}{\mathrm{d}x^{(i)}} \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u^{(l)}\left(r, X_r^{(1)}, X_r^{(2)}\right) = \frac{\mathrm{d}}{\mathrm{d}x^{(i)}} Y_r^{(l1)}$ for all $l = 1, 2$,
- $Y^{(l_1 l_2)}$, $Y^{(l)}$, $Z^{(l)}$ are always uniformly bounded as was already mentioned,
- $Z^{(jk)}$, $j, k = 1, 2$ have uniformly bounded $BMO(\mathbb{P})$ - norms according to the dynamics of $Y^{(jk)}$ and Theorem A.1.11.

Let $j, k \in \{1, 2\}$. As a consequence of the decoupling condition together with the chain rule of Lemma A.3.1:

$$\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} Y_r^{(jk)} = \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u^{(jk)}(r, X_r^{(1)}, X_r^{(2)}) + \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} u^{(jk)}(r, X_r^{(1)}, X_r^{(2)}) \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} X_r^{(2)},$$

$$\frac{\mathrm{d}}{\mathrm{d}x^{(2)}} Y_r^{(jk)} = \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} u^{(jk)}(r, X_r^{(1)}, X_r^{(2)}) \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} X_r^{(2)}.$$

Using the boundedness of $\left(\frac{\mathrm{d}}{\mathrm{d}x^{(2)}} X^{(2)}\right)^{-1}$ the second equation implies boundedness of

$$\frac{\mathrm{d}}{\mathrm{d}x^{(2)}} u^{(jk)}(t, x^{(1)}, x^{(2)})$$

for almost all $x^{(1)}, x^{(2)}$ by a uniform constant. Now, the first equation together with uniform boundedness of $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} X_r^{(2)}$ and $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} Y_r^{(jk)}$ implies uniform boundedness of $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u^{(jk)}$ as well.
Considering Lemma 4.2.29 we have a contradiction and the proof is complete. □

Lemma 4.3.10. *Now, assume that*

- $T = 1$,
- g, δ, g', δ', g'', δ'' *are all Lipschitz continuous,*
- g *is increasing and not constant.*

Then the Markovian decoupling field u from Lemma 4.3.1 fulfills the requirements of Theorem 4.3.6.

Lemma 4.3.11. *Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be twice weakly differentiable s.t. $\varphi(0) = 0$ and $\|\varphi''\|_\infty < \infty$. Then*

$$\left| \int_{\mathbb{R}} \varphi(\sigma \cdot z) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \,\mathrm{d}z \right| \leq \frac{1}{2}\sigma^2 \|\varphi''\|_\infty,$$

for all $\sigma \in [0, \infty)$.

Proof. Using weak differentiability of φ we can write for any $x \in \mathbb{R}$:

$$\begin{aligned}
\varphi(x) &= \int_0^1 \varphi'(sx) x \,\mathrm{d}s \\
&= x \int_0^1 \left(\varphi'(0) + \int_0^1 \varphi''(tsx) sx \,\mathrm{d}t \right) \mathrm{d}s = x\varphi'(0) + x^2 \int_0^1 s \int_0^1 \varphi''(tsx) \,\mathrm{d}t \,\mathrm{d}s,
\end{aligned}$$

so

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi(\sigma \cdot z) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \,\mathrm{d}z \\
&\qquad = \int_{\mathbb{R}} \sigma z \varphi'(0) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \,\mathrm{d}z + \int_{\mathbb{R}} \sigma^2 z^2 \left(\int_0^1 s \int_0^1 \varphi''(ts\sigma z) \,\mathrm{d}t \,\mathrm{d}s \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \,\mathrm{d}z.
\end{aligned}$$

The first summand clearly vanishes and we can finally estimate:

$$\begin{aligned}\left|\int_{\mathbb{R}} \varphi(\sigma\cdot z)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\,\mathrm{d}z\right| &\leq \sigma^2\int_{\mathbb{R}} z^2\int_0^1 s\int_0^1 |\varphi''(ts\sigma z)|\,\mathrm{d}t\,\mathrm{d}s\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\,\mathrm{d}z\\ &\leq \sigma^2\int_{\mathbb{R}} z^2\int_0^1 s\|\varphi''\|_\infty\,\mathrm{d}s\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\,\mathrm{d}z\\ &= \sigma^2\int_{\mathbb{R}} z^2\frac{1}{2}\|\varphi''\|_\infty\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\,\mathrm{d}z = \sigma^2\frac{\|\varphi''\|_\infty}{2}.\end{aligned}$$

□

Proof of Lemma 4.3.10. Let $\left(u^{(0)},u^{(1)},u^{(2)},u^{(11)},u^{(12)},u^{(21)},u^{(22)}\right)$ be the unique Markovian decoupling field to the problem (4.21) on $[0,T]$. We have $u^{(0)}=u$, $u^{(1)}=\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u$, etc. according to Theorem 4.3.9.

Let us show that $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u$ is Lipschitz continuous in the *first* component (which is time): Consider for a starting time $t\in[0,T]$ and initial condition $x\in\mathbb{R}^2$ the associated FBSDE (4.21) on $[t,1]$. Remember that

$$Y_s^{(1)}=\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u(s,X_s^{(1)},X_s^{(2)}),\quad s\in[t,1], \tag{4.22}$$

satisfies dynamics

$$Y_s^{(1)}=Y_t^{(1)}+\int_t^s\left(-2Z_r^{(0)}Y_r^{(2)}\right)Z_r^{(1)}\,\mathrm{d}r+\int_t^s Z_r^{(1)}\,\mathrm{d}W_r,\quad s\in[t,1], \tag{4.23}$$

where

- $Z^{(0)}=\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(0)}\left(\cdot,X^{(1)}_\cdot,X^{(2)}_\cdot\right)=Y^{(1)}$ a.e. according to Lemma 4.3.8, which is applicable since $\left(\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(i)}\right)_{i=1,2}=\left(u^{(i1)}\right)_{i=1,2}$ and $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(0)}=u^{(1)}$ are continuous on $[t,1]$ according to Remark 4.2.21,
- $Z^{(0)}=Y^{(1)}$ and $Y^{(2)}$ are bounded by $\left\|\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u\right\|_\infty$ and $\left\|\frac{\mathrm{d}}{\mathrm{d}x^{(2)}}u\right\|_\infty$,
- $Z^{(1)}=\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(1)}\left(\cdot,X^{(1)}_\cdot,X^{(2)}_\cdot\right)$ a.e. according to Lemma 4.3.8, which is applicable as already mentioned. So, $Z^{(1)}$ is bounded by $\left\|\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u^{(1)}\right\|_\infty$.

Let $s\in(t,1]$. Using triangle inequality:

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u(s,x)-\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u(t,x)\right| \leq &\left|\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u(s,x)-\mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u(s,X_s^{(1)},X_s^{(2)})\right]\right|\\ &+\left|\mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u(s,X_s^{(1)},X_s^{(2)})\right]-\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u(t,x)\right|.\end{aligned}$$

Applying triangle inequality a second time together with (4.22):

$$\begin{aligned}&\left|\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u(s,x)-\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u(t,x)\right|\\ &\quad\leq\left|\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u(s,x^{(1)},x^{(2)})-\mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u(s,X_s^{(1)},x^{(2)})\right]\right|\\ &\qquad+\left|\mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u(s,X_s^{(1)},x^{(2)})\right]-\mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}x^{(1)}}u(s,X_s^{(1)},X_s^{(2)})\right]\right|+\left|\mathbb{E}\left[Y_s^{(1)}-Y_t^{(1)}\right]\right|\end{aligned}$$

Let us now control the three summands on the right-hand-side separately:

First summand: Define

$$\varphi(z) := \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u(s, x^{(1)}, x^{(2)}) - \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u(s, x^{(1)} + z, x^{(2)}), \quad z \in \mathbb{R}$$

and note:

- $\left| \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u(s, x^{(1)}, x^{(2)}) - \mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u(s, X_s^{(1)}, x^{(2)}) \right] \right| = \left| \int_{\mathbb{R}} \varphi(\sqrt{s-t} z) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \, \mathrm{d}z \right|$, since
$$X_s^{(1)} = x^{(1)} + W_s - W_t \sim \mathcal{N}\left(x^{(1)}, s - t \right),$$
- φ is Lipschitz continuous with Lipschitz constant $L_{u^{(1)}}$, which is the Lipschitz constant of $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u = u^{(1)}$ w.r.t. the last two components,
- φ' is Lipschitz continuous with Lipschitz constant $L_{u^{(11)}}$, which is the Lipschitz constant of $\frac{\mathrm{d}^2}{\left(\mathrm{d}x^{(1)}\right)^2} u = u^{(11)}$ w.r.t. the last two components,
- $\varphi(0) = 0$.

And so using Lemma 4.3.11 we obtain

$$\left| \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u(s, x^{(1)}, x^{(2)}) - \mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u(s, X_s^{(1)}, x^{(2)}) \right] \right| \le \frac{1}{2}(s-t) \cdot L_{u^{(11)}}.$$

Second summand: We have

$$\begin{aligned} &\left| \mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u(s, X_s^{(1)}, x^{(2)}) \right] - \mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u(s, X_s^{(1)}, X_s^{(2)}) \right] \right| \\ &\qquad \le \mathbb{E}\left[\left| \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u(s, X_s^{(1)}, x^{(2)}) - \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u(s, X_s^{(1)}, X_s^{(2)}) \right| \right] \le L_{u^{(1)}} \mathbb{E}\left[\left| X_s^{(2)} - x^{(2)} \right| \right], \end{aligned}$$

while

$$\left| X_s^{(2)} - x^{(2)} \right| = \left| \int_t^s \left(Z_r^{(0)} \right)^2 \mathrm{d}r \right| \le (s-t) \cdot \left\| Y^{(1)} \right\|_\infty^2 \le (s-t) \cdot \left\| \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u \right\|_\infty^2 \quad \text{a.s.},$$

where we used $Z^{(0)} = Y^{(1)}$ a.e.

Third summand: We have using (4.23):

$$\begin{aligned} \left| \mathbb{E}\left[Y_s^{(1)} - Y_t^{(1)} \right] \right| &= \left| \mathbb{E}\left[-2 \int_t^s Y_r^{(1)} Y_r^{(2)} Z_r^{(1)} \, \mathrm{d}r \right] \right| \\ &\le 2 \cdot (s-t) \cdot \left\| \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u \right\|_\infty \cdot \left\| \frac{\mathrm{d}}{\mathrm{d}x^{(2)}} u \right\|_\infty \cdot \left\| \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u^{(1)} \right\|_\infty. \end{aligned}$$

Conclusion: We have demonstrated

$$\left| \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u(s, x) - \frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u(t, x) \right| \le C|s - t|,$$

with some constant $C \in [0, \infty)$, which does not depend on t, x or s. In other words $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u$ is Lipschitz continuous in time. Since it is also Lipschitz continuous in space, it is a Lipschitz continuous function on its whole domain $[0, T] \times \mathbb{R}^2$.

It remains to show that $\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u$ is $\mathbb{R} \backslash \{0\}$-valued on $[0, 1) \times \mathbb{R}^2$:

Clearly, g' is non-negative and does not vanish. Let $t \in [0,1)$, $x \in \mathbb{R}^2$. Consider the associated FBSDE on $[t,1]$. Using (4.23) we can write

$$\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u(t,x) = g'(X_T^{(1)}) - \int_t^T Z_r^{(1)} \,\mathrm{d}\left(W_r + \int_t^r \left(-2Y_\kappa^{(1)} Y_\kappa^{(2)}\right) \mathrm{d}\kappa\right)$$

and so there is a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u(t,x) = \mathbb{E}_{\mathbb{Q}}\left[g'\left(X_T^{(1)}\right)\right] \geq 0.$$

Now, note that $X_T^{(1)} = x^{(1)} + W_T - W_t$ has a non-degenerate normal distribution w.r.t. $\mathbb{P}$. Therefore, its distribution is equivalent to the Lebesgue measure, but since $\mathbb{Q} \sim \mathbb{P}$ the distribution of $X_T^{(1)}$ w.r.t. $\mathbb{Q}$ must also be equivalent to the Lebesgue measure. This shows

$$\frac{\mathrm{d}}{\mathrm{d}x^{(1)}} u(t,x) = \mathbb{E}_{\mathbb{Q}}\left[g'\left(X_T^{(1)}\right)\right] > 0$$

since otherwise $g' = 0$ a.e. would hold. $\square$

Chapter 5

Solving FBSDE Arising in Problems of Utility Maximization in Incomplete Markets

The central problem to which we apply techniques of forward-backward stochastic differential equations (FBSDE) in this chapter originally comes from *securitization*, i.e. insuring market exogenous risk by investing on a capital market. Typically, a small agent whose preferences are described by a utility function U wants to insure a random liability H arising from his usual business. He, therefore, has two sources of income: his random liability, and the wealth obtained from trading on the capital market up to a terminal time with appropriate investment strategies. The *stochastic control problem* he faces results in the maximization of his terminal utility obtained from both sources of income with respect to all admissible strategies. More formally, given his initial wealth $x > 0$, he wants to attain the value

$$V(0, x) := \sup_{\pi \in \mathcal{A}} \mathbb{E}[U(X_T^\pi + H)], \tag{5.1}$$

where $\mathcal{A}$ is the set of *admissible trading strategies*, $T < \infty$ the trading horizon, X_T^π the agent's terminal wealth related to an investment strategy $\pi \in \mathcal{A}$. This wealth is obtained from investing in a financial market composed of a zero interest rate bond, and $d \geq 1$ stocks given by

$$dS_t^i := S_t^i dW_t^i + S_t^i \theta_t^i dt, \quad i \in \{1, \ldots, d\},$$

where W is a standard Brownian motion on $\mathbb{R}^d$, and θ the process describing market risk. As in [HHI$^+$14], trading underlies a linear constraint: assume $d_1 + d_2 = d$ and that the agent can only invest in the assets $S^1, \ldots, S^{d_1}$. Other relevant problems in this context are the characterization of optimal strategies and the *value function* V which for $0 \leq t \leq T$ is given by

$$V(t, x) := \sup_{\pi \in \mathcal{A}} \mathbb{E}[U(X_{t,T}^\pi + H) | \mathcal{F}_t].$$

Here $X_{t,T}$ is the wealth obtained in the investment period $[t, T]$, and $(\mathcal{F}_t)_{t \in [0,T]}$ describes the evolution of information.

The most common technique employed to obtain optimal strategies π^* is related to *convex analysis and duality* (see Bismut [Bis76], Pliska [Pli86], Karatzas and co-workers (Karatzas et al. [KLS87], [KLSX91], [CK92]), Kramkov and Schachermayer [KS99]). A direct stochastic approach to characterize optimal trading strategies in the case of non-linear, even non-convex trading constraints is provided by an interpretation of the martingale optimality principle by (forward) backward stochastic differential equations (FBSDE) (see El Karoui et al. [REK00], Sekine [Sek06], and Hu et al. [HIM05]). If the utility function is exponential, or power or logarithmic (and $H = 0$), it has been shown by Hu et al. [HIM05] that the control problem (5.1) can essentially be reduced to solving a BSDE of the form

$$Y_t = H - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds, \quad t \in [0, T], \tag{5.2}$$

where the *driver* $f(t, z)$ is of quadratic growth in the z-variable. While for these classical utility functions forward and backward components of the investment dynamics decouple, in [HHI$^+$14] the problem (5.1) has been tackled for a larger class of utility functions, and shown to lead to a fully-coupled system of FBSDE, again typically with a driver of quadratic growth in the control variable. The derivation of this system starts with a verification type observation. Given an optimal strategy π^* of the (forward) portfolio process X^{π^*}, to realize martingale optimality one postulates that $U'(X^{\pi^*}+Y)$ is a martingale, where (Y, Z) is the associated backward process. As a consequence, (Y, Z) is given by a certainty equivalent type expression for marginal utility $Y = (U')^{-1}(\mathbb{E}(U'(X_T^{\pi^*} + H)|\mathcal{F}_\cdot) - X^{\pi^*}$. This allows to compute the driver of the BSDE related to (Y, Z). It is given in terms of the derivatives of U, involves the optimal forward process X^{π^*}, and provides the backward part of the FBSDE system. In a second step, one considers possible solution triples (X, Y, Z) of the FBSDE system obtained in the first step, not assuming that X corresponds to an optimal portfolio process. One then uses the variational maximum principle in order to verify that under mild conditions on U the triple (X, Y, Z) solves the original optimization problem. This in particular means that X coincides with an optimal forward portfolio process X^{π^*}. In summary, under mild regularity conditions, solutions (X, Y, Z) of the FBSDE system provide solutions of the original securitization problem. If θ is the price of risk process associated to the price dynamics of the market, and π_1 denotes the projection on the first d_1 coordinates in $\mathbb{R}^d$, π_2 the one on the remaining d_2 coordinates, π^* is given by

$$\pi^* = -\pi_1(\theta)\frac{U'(X + Y)}{U''(X + Y)} - \pi_1(Z), \tag{5.3}$$

and the FBSDE by

$$\begin{aligned} X_t &= x - \int_0^t \left(\pi_1(\theta_s)\frac{U'(X_s + Y_s)}{U''(X_s + Y_s)} + \pi_1(Z_s)\right)^\top \mathrm{d}W_s - \int_0^t \left(\pi_1(\theta_s)\frac{U'(X_s + Y_s)}{U''(X_s + Y_s)} + \pi_1(Z_s)\right)^\top \pi_1(\theta_s)\,\mathrm{d}s, \\ Y_t &= H - \int_t^T Z_s^\top \,\mathrm{d}W_s - \int_t^T \left[-\frac{1}{2}|\pi_1(\theta_s)|^2 \frac{U^{(3)}(X_s + Y_s)|U'(X_s + Y_s)|^2}{(U''(X_s + Y_s))^3} + \right. \\ &\qquad \left. + |\pi_1(\theta_s)|^2 \frac{U'(X_s + Y_s)}{U''(X_s + Y_s)} + Z_s \cdot \pi_1(\theta_s) - \frac{1}{2}|\pi_2(Z_s)|^2 \frac{U^{(3)}}{U''}(X_s + Y_s)\right] \mathrm{d}s. \end{aligned} \tag{5.4}$$

This of course only translates the original utility optimization problem into another problem the solvability of which is far from obvious.

In this chapter we use the technique of decoupling fields to find a framework in which solution triples of systems as the one above exist and are unique. To sketch the tool of decoupling fields we apply in this context, consider a general FBSDE system of the form

$$X_t = x + \int_0^t \mu(s, X_s, Y_s, Z_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s, Y_s, Z_s)\,\mathrm{d}W_s,$$

$$Y_t = \xi(X_T) - \int_t^T f(s, X_s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s.$$

X and Y may be multidimensional, and the particular nature of the underlying problem is encoded in the parameter functions μ, σ, f which may be random, but at least progressively measurable. The terminal condition ξ, besides the terminal value of the forward process X, may have a further dependence on randomness, and is required to be measurable w.r.t. $\mathcal{F}_T$.

In this chapter we apply the general results obtained in Chapter 2 to the particular setting of (5.4) in order to obtain existence and uniqueness of solutions. This requires some conditions concerning the structure of the utility functions considered, expressed by boundedness conditions for quotients of its derivatives up to order 3 together with the condition $(\ln(-U''))'' \leq 0$ which we will motivate in Remark 5.2.4. The extension of existence and regularity results for decoupling fields to the situation of FBSDE generators with quadratic growth also leads us to consider a (Markovian) scenario in which the market price of risk process together with the terminal condition ξ may depend on randomness, but only through the values of an external standard, possibly high-dimensional diffusion. In this scenario, the case in which the driver is only locally Lipschitz is reduced to the Lipschitz case by obtaining effective bounds on the control process through its description by the decoupling field and its derivatives.

The chapter is organized as follows. In section 1 we recall basic notions and facts about the tool of decoupling fields from Chapter 2. In Section 5.2 we present the most important results about existence, uniqueness and structure of solutions of decoupling fields for coupled FBSDE for the case of Lipschitz continuous drivers. In particular, conditions on the coefficients of the FBSDE system will imply the well-posedness of the FBSDE, i.e. lead to a solution on the entire time interval $[0, T]$. In Theorem 5.2.2, under the aforementioned conditions on U, and for liabilities H that may depend on the entire history of the price dynamics, we derive a unique solution for a truncated version of (5.4).

Section 5.3 is devoted to the treatment of the scenario in which randomness in the price dynamics and terminal condition is mediated by an exogenous standard diffusion, which takes us into a type of Markovian setting. In the main result, Theorem 5.3.13, again certain structure conditions on the utility function are needed, to guarantee the existence of a unique solution of (5.4).

5.1 The method of decoupling fields

We consider families (μ, σ, f) of measurable functions, more precisely

$$\mu : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \longrightarrow \mathbb{R}^n,$$
$$\sigma : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \longrightarrow \mathbb{R}^{n \times d},$$
$$f : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \longrightarrow \mathbb{R}^m,$$

where

- $n, m, d \in \mathbb{N}$ and $T > 0$,
- $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]})$ is a complete filtered probability space,
- $\mathcal{F}_0 = \sigma(p) \vee \mathcal{N}$ for some $p : \Omega \to S$ and a polish space S, $\mathcal{N}$ are the null sets,
- $\mathcal{F}_t = \sigma(\mathcal{F}_0, (W_s)_{s \in [0,t]})$ holds, where $(W_t)_{t \in [0,T]}$ is a d-dimensional Brownian motion, independent of $\mathcal{F}_0$,
- $\mathcal{F} = \mathcal{F}_T$.

We want μ, σ and f to be progressively measurable w.r.t. $(\mathcal{F}_t)_{t \in [0,T]}$, i.e. $\mu \mathbf{1}_{[0,t]}, \sigma \mathbf{1}_{[0,t]}, f \mathbf{1}_{[0,t]}$ must be $\mathcal{B}([0, T]) \otimes \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^{m \times d})$ - measurable for all $t \in [0, T]$. We will assume throughout the chapter that μ, σ and f have this property without mentioning it.

Definition 5.1.1. Let $\xi : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ be measurable and let $t \in [0, T]$.
We call a function $u : [t, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ with $u(T, \omega, \cdot) = \xi(\omega, \cdot)$ for a.a. $\omega \in \Omega$ a *decoupling field* for $(\xi, (\mu, \sigma, f))$ on $[t, T]$ if for all $t_1, t_2 \in [t, T]$ with $t_1 \leq t_2$ and any $\mathcal{F}_{t_1}$ - measurable $X_{t_1} : \Omega \to \mathbb{R}^n$ there exist progressive processes X, Y, Z on $[t_1, t_2]$ such that

- $X_s = X_{t_1} + \int_{t_1}^s \mu(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_{t_1}^s \sigma(r, X_r, Y_r, Z_r)\,\mathrm{d}W_r$ a.s.,
- $Y_s = Y_{t_2} - \int_s^{t_2} f(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^{t_2} Z_r\,\mathrm{d}W_r$ a.s.,
- $Y_s = u(s, X_s)$ a.s.,

for all $s \in [t_1, t_2]$. In particular, we want all integrals to be well-defined and X, Y, Z to have values in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$ respectively.

Some remarks about this definition:

- In the above definition the first equation is called the *forward equation*, the second the *backward equation* and the third will be referred to as the *decoupling condition*.
- This requirement that X should start at X_{t_1} is referred to as the *initial condition*. By a slight abuse of notation we will sometimes refer to X_{t_1} itself as the initial condition. X_{t_1} is also sometimes referred to as the *initial value*.
- Note that if $t_2 = T$, we get $Y_T = \xi(X_T)$ a.s. as a consequence of the decoupling condition together with $u(T, \omega, \cdot) = \xi(\omega, \cdot)$ for a.a. $\omega \in \Omega$.
- If $t_2 = T$, we can say that a triple (X, Y, Z) solves the FBSDE, meaning that it satisfies the forward and the backward equation, together with $Y_T = \xi(X_T)$. This relationship $Y_T = \xi(X_T)$ is referred to as the *terminal condition*.

Decoupling fields have the following very important property, which distinguishes them from classical solutions to FBSDEs.

Lemma 5.1.2 (Lemma 2.1.2 in Chapter 2)**.** *If u is a decoupling field for $(\xi, (\mu, \sigma, f))$ on $[t, T]$ and a map $\tilde{u}$ is a decoupling field for $(u(t, \cdot), (\mu, \sigma, f))$ on $[s, t]$, where $0 \le s < t < T$, then the map*

$$\hat{u} := \tilde{u}\mathbf{1}_{[s,t]} + u\mathbf{1}_{(t,T]}$$

is a decoupling field for $(\xi, (\mu, \sigma, f))$ on $[s, T]$.

Note that according to definition if u is a decoupling field and $\tilde{u}$ is a modification of u, i.e. for each $s \in [t, T]$ the functions $u(s, \omega, \cdot)$ and $\tilde{u}(s, \omega, \cdot)$ coincide for a.a. $\omega \in \Omega$, then $\tilde{u}$ is also a decoupling field to the same problem. So, u could also be referred to as a class of modifications. Some of the members of the class might be progressively measurable, others not. However one can show that a progressively measurable representative does exist if the decoupling field is Lipschitz continuous in x (Lemma 2.1.3 in Chapter 2).

Let $I \subseteq [0, T]$ be an interval and $u : I \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ a map such that $u(s, \cdot)$ is measurable for every $s \in I$. We define

$$L_{u,x} := \sup_{s\in I} \inf\{L \ge 0 \,|\, \text{for a.a. } \omega \in \Omega : |u(s, \omega, x) - u(s, \omega, x')| \le L|x - x'| \text{ for all } x, x' \in \mathbb{R}^n\}.$$

where $\inf \emptyset := \infty$. We also set $L_{u,x} := \infty$ if $u(s, \cdot)$ is not measurable for every $s \in I$. One can show that $L_{u,x} < \infty$ is equivalent to u having a modification, which is truly Lipschitz continuous in $x \in \mathbb{R}^n$.

We denote by $L_{\sigma,z}$ the Lipschitz constant of σ w.r.t. its dependence on the last component z (and w.r.t. the Frobenius norms on $\mathbb{R}^{m\times d}$ and $\mathbb{R}^{n\times d}$). We set $L_{\sigma,z} = \infty$ if σ is not Lipschitz continuous in z.

By $L_{\sigma,z}^{-1} = \frac{1}{L_{\sigma,z}}$ we mean $\frac{1}{L_{\sigma,z}}$ if $L_{\sigma,z} > 0$ and ∞ otherwise.

Definition 5.1.3. Let $u : [t, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ be a decoupling field to $(\xi, (\mu, \sigma, f))$. We call u *weakly regular*, if $L_{u,x} < L_{\sigma,z}^{-1}$ and $\sup_{s\in[t,T]} \|u(s, \cdot, 0)\|_\infty < \infty$.

Weak regularity implies weak differentiability of u w.r.t. x. It also allows to assume that u has a progressively measurable modification. In practice, however, it is important to have explicit knowledge about the regularity of (X, Y, Z). For instance, it is important to know in which spaces the processes live, and how they react to changes in the initial value. Specifically it can be very useful to have differentiability of X, Y, Z w.r.t. the initial value.

In the following we need further notation. For an integrable real valued random variable X the expression $\mathbb{E}_t[X]$ refers to $\mathbb{E}[X|\mathcal{F}_t]$, while $\mathbb{E}_{\hat{t},\infty}[X]$ refers to $\operatorname{ess\,sup}\mathbb{E}[X|\mathcal{F}_t]$, which might be ∞, but is always well-defined as the infimum of all constants $c \in [-\infty, \infty]$ such that $\mathbb{E}[X|\mathcal{F}_t] \leq c$ a.s..
As usual $\|X\|_\infty$ refers to the essential supremum of $|X|$, for an arbitrary measurable X.

Definition 5.1.4. Let $u : [t, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ be a weakly regular decoupling field to $(\xi, (\mu, \sigma, f))$. We call u *strongly regular* if for all fixed $t_1, t_2 \in [t, T]$, $t_1 \leq t_2$, the processes X, Y, Z arising in the defining property of a decoupling field are a.e. unique for each *constant* initial value $X_{t_1} = x \in \mathbb{R}^n$ and satisfy

$$\sup_{s\in[t_1,t_2]} \mathbb{E}_{t_1,\infty}[|X_s|^2] + \sup_{s\in[t_1,t_2]} \mathbb{E}_{t_1,\infty}[|Y_s|^2] + \mathbb{E}_{t_1,\infty}\left[\int_{t_1}^{t_2} |Z_s|^2 \,\mathrm{d}s\right] < \infty \quad \forall x \in \mathbb{R}^n. \tag{5.5}$$

In addition they must be measurable as functions of (x, s, ω) and even weakly differentiable w.r.t. $x \in \mathbb{R}^n$ such that for every $s \in [t_1, t_2]$ the mappings X_s and Y_s are measurable functions of (x, ω) and even weakly differentiable w.r.t. x such that

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \sup_{s\in[t_1,t_2]} \mathbb{E}_{t_1,\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}x}X_s\right|_v^2\right] < \infty,$$

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \sup_{s\in[t_1,t_2]} \mathbb{E}_{t_1,\infty}\left[\left|\frac{\mathrm{d}}{\mathrm{d}x}Y_s\right|_v^2\right] < \infty,$$

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \mathbb{E}_{t_1,\infty}\left[\int_{t_1}^{t_2}\left|\frac{\mathrm{d}}{\mathrm{d}x}Z_s\right|_v^2 \mathrm{d}s\right] < \infty. \tag{5.6}$$

We say that a decoupling field u on $[t, T]$ is *strongly regular* on a subinterval $[t_1, t_2] \subseteq [t, T]$ if u restricted to $[t_1, t_2]$ is a strongly regular decoupling field for $(u(t_2, \cdot), (\mu, \sigma, f))$.

Strong regularity is a fundamental concept in our theory. It allows to work with weak derivatives and apply the rules of Lemmas A.2.4 to A.2.8 in particular. Consult Section 2.1.2 of Chapter 2 for more on the subject of weak derivatives.

Under certain conditions a rich existence, uniqueness and regularity theory for decoupling fields can be developed. We will summarize the main results, which are proven in Chapter 2, Section 2.5:

Definition 5.1.5. We say that ξ, μ, σ, f satisfy *standard Lipschitz conditions (SLC)* if

- μ, σ, f are Lipschitz continuous in (x, y, z) with some Lipschitz constant L,
- $\|(|\mu| + |f| + |\sigma|)(\cdot, \cdot, 0, 0, 0)\|_\infty < \infty$,
- $\xi : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ is measurable s.t. $\|\xi(\cdot, 0)\|_\infty < \infty$ and $L_{\xi,x} < L_{\sigma,z}^{-1}$.

Theorem 5.1.6 (Global uniqueness). *Let μ, σ, f, ξ satisfy SLC and assume that there are two weakly regular decoupling fields $u^{(1)}, u^{(2)}$ to the corresponding problem on some interval $[t, T]$. Then $u^{(1)} = u^{(2)}$ up to modifications.*

Proof. Corollary 2.5.3 in Chapter 2. □

Theorem 5.1.7 (Global regularity). *Let μ, σ, f, ξ satisfy SLC and assume that there exists a weakly regular decoupling field u to this problem on some interval $[t, T]$. Then u is strongly regular.*

Proof. Corollary 2.5.4 in Chapter 2. □

Notice that Theorem 5.1.6 only provides uniqueness of weakly regular decoupling fields, not uniqueness of processes (X, Y, Z) solving the FBSDE in the classical sense. However, using Theorem 5.1.7 one can show:

Corollary 5.1.8. *Let μ, σ, f, ξ satisfy SLC and assume that there exists a weakly regular decoupling field u on some interval $[t, T]$.*
Then for any initial condition $X_t = x \in \mathbb{R}^n$ there is a unique solution (X, Y, Z) of the FBSDE on $[t, T]$ satisfying

$$\sup_{s\in[t,T]} \mathbb{E}_{0,\infty}[|X_s|^2] + \sup_{s\in[t,T]} \mathbb{E}_{0,\infty}[|Y_s|^2] + \mathbb{E}_{0,\infty}\left[\int_t^T |Z_s|^2 \, \mathrm{d}s\right] < \infty. \tag{5.7}$$

Proof. Corollary 2.5.5 in Chapter 2. □

Now, we want to investigate how large the interval $[t, T]$ can be chosen, such that we still have (weakly regular) decoupling fields on this interval. It is natural to work with the following definition.

Definition 5.1.9. We define the *maximal interval* $I_{\max} \subseteq [0, T]$ for $(\xi, (\mu, \sigma, f))$ as the union of all intervals $[t, T] \subseteq [0, T]$, such that there exists a weakly regular decoupling field u on $[t, T]$.

Unfortunately the maximal interval might very well be open to the left. Therefore, we need to make our notions more precise in the following definitions.

Definition 5.1.10. Let $t < T$. We call a function $u : (t, T] \times \mathbb{R}^n \to \mathbb{R}^m$ a decoupling field for $(\xi, (\mu, \sigma, f))$ on $(t, T]$ if u restricted to $[t', T]$ is a decoupling field for all $t' \in (t, T]$.
We call a decoupling field u on $(t, T]$ *weakly regular* if u restricted to $[t', T]$ is weakly regular for all $t' \in (t, T]$. Similarly we call a decoupling field u on $(t, T]$ *strongly regular* if u restricted to $[t', T]$ is strongly regular for all $t' \in (t, T]$.

Now, we can formulate

Theorem 5.1.11 (Global existence in weak form). *Let μ, σ, f, ξ satisfy SLC. Then there exists a unique weakly regular decoupling field u on $I_{\max}$. This u is even strongly regular.*
Furthermore, either $I_{\max} = [0, T]$ or $I_{\max} = (t_{\min}, T]$ where $0 \leq t_{\min} < T$.

Proof. Theorem 2.5.11 in Chapter 2. □

5.1.1 Global existence in strong form

By *global existence in strong form* we mean the above weak global existence together with $I_{\max} = [0, T]$. Unfortunately the "bad" case $I_{\max} = (t_{\min}, T]$ is possible and is even more common. The following result basically says that this case can only occur if there is an "explosion" in the spatial derivative of u as we approach the lower boundary $t_{\min}$. By "explosion" we mean reaching the "forbidden" value $L_{\sigma,z}^{-1}$ which is equal ∞ in many applications.

Lemma 5.1.12 (Lemma 2.5.12 in Chapter 2). *Let μ, σ, f, ξ satisfy SLC. If $I_{\max} = (t_{\min}, T]$, then*

$$\lim_{t\downarrow t_{\min}} L_{u(t,\cdot),x} = L_{\sigma,z}^{-1},$$

where u is the unique decoupling field on $I_{\max}$.

Lemma 5.1.12 serves as a blueprint to show strong global existence in those cases in which it is suspected to hold. Let us describe the different steps.

1. Assume indirectly that $I_{\max} = [0,T]$ does not hold, which implies $I_{\max} = (t_{\min}, T]$. Choose arbitrary $t \in (t_{\min}, T]$, $x \in \mathbb{R}^n$ and consider the corresponding FBSDE.

2. Differentiate the FBSDE w.r.t. x. This is possible because of strong regularity of u (Theorem 5.1.11). We obtain joint dynamics of $\frac{\mathrm{d}}{\mathrm{d}x}X$, $\frac{\mathrm{d}}{\mathrm{d}x}Y$, $\frac{\mathrm{d}}{\mathrm{d}x}Z$.

3. Using Itô's formula deduce the dynamics of $\frac{\mathrm{d}}{\mathrm{d}x}Y_s(\frac{\mathrm{d}}{\mathrm{d}x}X_s)^{-1}$. Note that this process should be equal to $u_x(s, X_s)$, as a consequence of the decoupling condition $Y_s = u(s, X_s)$.

4. Using the dynamics of $u_x(s, X_s)$ show that its modulus can be bounded away from $L_{\sigma,z}^{-1}$ independently of t, x, s, ω. This contradicts Lemma 5.1.12 and, therefore, $I_{\max} = [0,T]$ must hold.

This blueprint can be referred to as the *method of decoupling fields* to show global existence and uniqueness of solutions to FBSDEs (note Corollary 5.1.8 at this point). Steps 1., 2. and 3. can be done in a rather general setting. Step 4., however, seems to be much more problem specific.

For later reference note the chain rules of Lemmas A.3.1 and A.3.2.

5.2 Main results for the Lipschitz case

The main goal of this section is to show Theorem 5.2.2, which essentially states that a truncated version of FBSDE (5.4) has a unique solution under certain conditions on U. The structure of U implied by these conditions is discussed in Remark 5.2.4. The truncation occurs in Z in order to create Lipschitz continuity. We first prove Theorem 5.2.1 which is an abstract result showing well-posedness for a class of FBSDE satisfying certain structural properties. Theorem 5.2.2 will then be a rather direct consequence of Theorem 5.2.1. Due to truncation the trading strategies obtained from the new FBSDE will not be necessarily optimal under the probability measure $\mathbb{P}$, but under some equivalent probability measure "infinitesimally close" to it as Lemma 5.2.3 shows.

We use the notation of the previous section. In particular, remember the meaning of $n, m, d \in \mathbb{N}$. For the following assume we have $d_1 \in \mathbb{N}$, $d_2 \in \mathbb{N}_0$ such that $d_1 + d_2 = d$, which is the dimension of our Brownian motion.
For a vector $z \in \mathbb{R}^d$ we denote by $\pi_1(z) \in \mathbb{R}^d$ a vector whose first d_1 components coincide with the first d_1 components of z and the remaining are set to 0.
Similarly we define $\pi_2(z)$ as the vector, where the first d_1 components are set to 0 and the rest remains unchanged.
We can define mappings π_1, π_2 for row vectors from $\mathbb{R}^{1\times d}$ or even from $\mathbb{R}^{1\times(1\times d)}$ in an analogous way as well.
If we assume $n = 1$, then σ is a row vector of dimension d. The expression σ_j, where $j = 1, \ldots, d$ refers to the components of this vector. As usual they depend on (ω, t, x, y, z).
We now formulate and prove the following theorem:

Theorem 5.2.1. *Let μ, σ, f, ξ satisfy SLC and assume furthermore:*

- $n = m = 1$,
- *μ is merely a function of ω, s,*
- *f is a function of $\omega, s, x + y, \pi_2(z)$ s.t.* $\frac{\mathrm{d}}{\mathrm{d}(x+y)}f = \frac{\mathrm{d}}{\mathrm{d}x}f = \frac{\mathrm{d}}{\mathrm{d}y}f \geq 0$ *a.e.,*

- $\sigma = \left(\sigma^{(1)}, \sigma^{(2)}\right)$, *with* d_1 *- dimensional* $\sigma^{(1)}$ *and* d_2 *- dimensional* $\sigma^{(2)}$, *such that* $\sigma^{(1)}$ *is a function of* $\omega, s, x, y, \pi_1(z)$ *and* $\sigma^{(2)}$ *is a function of* $\omega, s, \pi_2(z)$, *s.t.* $L_{\sigma,z} \leq 1$.

Then for this problem $I_{\max} = [0, T]$ *will hold.*

Proof. We conduct an indirect proof: Assume $I_{\max} = [0, T]$ does not hold, i.e. assume $I_{\max} = (t_{\min}, T]$. Let u be the strongly regular decoupling field from Theorem 5.1.11. Choose any $t_1 \in (t_{\min}, T]$, any $x \in \mathbb{R}$ and consider the corresponding X, Y, Z on $[t_1, T]$ such that

- $X_s = x + \int_{t_1}^s \mu(r)\,\mathrm{d}r + \int_{t_1}^s \sigma(r, X_r, Y_r, Z_r)\,\mathrm{d}W_r$,
- $Y_s = \xi(X_T) - \int_s^T f(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^T Z_r\,\mathrm{d}W_r$ and
- $Y_s = u(s, X_s)$ a.s.

for all $s \in [t_1, T]$.
We differentiate w.r.t. x using strong regularity and Lemma A.3.2:

$$\frac{\mathrm{d}}{\mathrm{d}x} X_s = 1 + \int_{t_1}^s \left(\delta^{\sigma,x} \frac{\mathrm{d}}{\mathrm{d}x} X_r + \delta_r^{\sigma,y} \frac{\mathrm{d}}{\mathrm{d}x} Y_r + \delta_r^{\sigma,z} \frac{\mathrm{d}}{\mathrm{d}x} Z_r\right) \mathrm{d}W_r,$$

$$\frac{\mathrm{d}}{\mathrm{d}x} Y_s = \frac{\mathrm{d}}{\mathrm{d}x} Y_T - \int_s^T \frac{\mathrm{d}}{\mathrm{d}x} Z_r\,\mathrm{d}W_r - \int_s^T \left(\delta_r^{f,x+y} \frac{\mathrm{d}}{\mathrm{d}x} X_r + \delta_r^{f,x+y} \frac{\mathrm{d}}{\mathrm{d}x} Y_r + \delta_r^{f,z} \frac{\mathrm{d}}{\mathrm{d}x} Z_r\right) \mathrm{d}r$$

a.s. for every $s \in [t_1, T]$, for a.a. $x \in \mathbb{R}$. The objects $\delta^{\sigma,x}$, $\delta^{\sigma,y}$, $\delta^{\sigma,z}$, $\delta^{f,x+y}$, $\delta^{f,z}$ are progressively measurable and uniformly bounded processes. Their boundedness is a consequence of the Lipschitz continuity of σ, f. Note also that according to Lemma A.3.2 and the structural assumptions on σ, f we made, these processes can be chosen in such a way that:

- $\pi_1\left(\delta^{f,z}\right)$ vanishes, whereas $\pi_2\left(\delta^{f,z}\right)$ is bounded by $L_{f,\pi_2(z)}$, which is the Lipschitz constant of f w.r.t. $\pi_2(z)$,
- $\delta^{f,x+y}$ is a non-negative real-valued process bounded by $L_{f,x+y}$, which is the Lipschitz constant of f w.r.t. $x + y$,
- $\delta^{\sigma,z}$ is $\mathbb{R}^{(1\times d)\times(1\times d)}$ - valued and can also be interpreted as a bounded matrix consisting of an upper left $d_1 \times d_1$ block and a lower right $d_2 \times d_2$ block, such that all remaining components vanish and the operator norm of the matrix itself is bounded by $L_{\sigma,z}$,
- $\delta^{\sigma,x}$ is $\mathbb{R}^{1\times d}$ - valued and can also be interpreted as a vector $(\delta^{\sigma_i,x})_{i=1,\ldots,d}$, where the last d_2 components vanish,
- $\delta^{\sigma,y}$ is $\mathbb{R}^{1\times d}$ - valued and can also be interpreted as a vector $(\delta^{\sigma_i,y})_{i=1,\ldots,d}$, where the last d_2 components vanish.

Now, define $U_r := \frac{\mathrm{d}}{\mathrm{d}x} X_r$ and $V_r := \frac{\mathrm{d}}{\mathrm{d}x} Y_r$ and $\tilde{Z}_r := \frac{\mathrm{d}}{\mathrm{d}x} Z_r$ for short. Note that U and V are $\mathbb{R}$ - valued and $\tilde{Z}$ is $\mathbb{R}^{1\times d}$ - valued. So, we have a linear FBSDE

$$U_s = 1 + \int_{t_1}^s \left(\delta_r^{\sigma,x} U_r + \delta_r^{\sigma,y} V_r + \delta_r^{\sigma,z} \tilde{Z}_r\right) \mathrm{d}W_r, \tag{5.8}$$

$$V_s = V_T - \int_s^T \tilde{Z}_r\,\mathrm{d}W_r - \int_s^T \left(\delta_r^{f,x+y} U_r + \delta_r^{f,x+y} V_r + \delta_r^{f,z} \tilde{Z}_r\right) \mathrm{d}r.$$

In particular, we can assume without loss of generality that U and V are continuous in time. Define a process $\hat{V}$ via $\hat{V}_r := \frac{\mathrm{d}}{\mathrm{d}x} u(r, X_r)$, $r \in [t_1, T]$. We can assume without loss of generality that $\hat{V}$ is

bounded, since u is Lipschitz continuous in x on every compact interval and we can assume that $\frac{\mathrm{d}}{\mathrm{d}x}u(t,\cdot)$ is uniformly bounded by $L_{u(t,\cdot),x}$ for every $t \in I_{\max}$.
Let $\tau \in [t_1, T]$ be any stopping time such that U is positive on $[t_1, \tau]$. We will argue later that we can choose $\tau = T$.
For $s \in [t_1, \tau]$ we have, using the chain rule of Lemma A.3.1, the relationship $V_s = \hat{V}_s U_s$. In particular, $\hat{V}_s \mathbf{1}_{[t_1,\tau]} = \frac{V_s}{U_s}\mathbf{1}_{[t_1,\tau]}$ and $\hat{V}$ has a modification, which is continuous on $[t_1, \tau]$. Defining $\hat{U} := U^{-1} = \frac{1}{U}$ on $[t_1, \tau]$, we can rewrite (5.8) as

$$U_{s\wedge\tau} = 1 + \int_{t_1}^{s\wedge\tau} \left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r\right) U_r \,\mathrm{d}W_r.$$

Applying the Itô formula to $\hat{U} = U^{-1} = \frac{1}{U}$ we obtain

$$\begin{aligned}\hat{U}_{s\wedge\tau} = 1 + \int_{t_1}^{s\wedge\tau} \hat{U}_r \bigg(\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r\right) \left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r\right)^\top \bigg) \,\mathrm{d}r - \\ - \int_{t_1}^{s\wedge\tau} \hat{U}_r \left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r\right) \mathrm{d}W_r.\end{aligned}$$

Applying the Itô formula to $\hat{V} = V\hat{U}$ we get

$$\begin{aligned}\hat{V}_{s\wedge\tau} = \hat{V}_\tau - \int_{s\wedge\tau}^{\tau} \tilde{Z}_r\hat{U}_r - \hat{V}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r\right) \mathrm{d}W_r - \\ - \int_{s\wedge\tau}^{\tau} \bigg(\delta_r^{f,x+y} + \delta_r^{f,x+y}\hat{V}_r + \delta_r^{f,z}\tilde{Z}_r\hat{U}_r + \\ + \hat{V}_r \bigg(\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r \right) \left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r\right)^\top \bigg) - \\ - \tilde{Z}_r\hat{U}_r \left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r\right)^\top \bigg) \,\mathrm{d}r.\end{aligned}$$

Note the two marked terms above. Define a process $\hat{Z}$ on $[t_1, \tau]$ via

$$\hat{Z}_r := \tilde{Z}_r\hat{U}_r - \hat{V}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r\right), \quad r \in [t_1, \tau].$$

Then we get

$$\begin{aligned}\hat{V}_{s\wedge\tau} = \hat{V}_\tau - \int_{s\wedge\tau}^{\tau} \hat{Z}_r \,\mathrm{d}W_r - \int_{s\wedge\tau}^{\tau} \bigg(\delta_r^{f,x+y} + \delta_r^{f,x+y}\hat{V}_r + \delta_r^{f,z}\tilde{Z}_r\hat{U}_r - \\ - \hat{Z}_r \left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r\right)^\top \bigg) \,\mathrm{d}r. \quad (5.9)\end{aligned}$$

Note the terms $\delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r$ and $\delta_r^{f,z}\tilde{Z}_r\hat{U}_r$. An easy calculation starting from the definition of $\hat{Z}_r$ leads to

$$\hat{Z}_r + \hat{V}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r\right) = \tilde{Z}_r\hat{U}_r - \hat{V}_r\delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r = \left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)\tilde{Z}_r\hat{U}_r,$$

$$\tilde{Z}_r\hat{U}_r = \left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)^{-1}\left(\hat{Z}_r + \hat{V}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r\right)\right), \quad r \in [t_1, \tau], \qquad (5.10)$$

where $I_d \in \mathbb{R}^{(1\times d)\times(1\times d)}$ is the identity.
Note here that $\left\|\hat{V}\right\|_\infty \leq L_{u_{|[t_1,T]},x} < L_{\sigma,z}^{-1}$ and also that the operator norm of $\delta^{\sigma,z}$ is universally bounded by $L_{\sigma,z}$, so the essential supremum of the operator norm of $\hat{V}_r\delta_r^{\sigma,z}$ is strictly smaller than 1

and, therefore, the expression $\left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)^{-1}$ is well-defined and even universally bounded (on $[t_1, T]$) in the operator norm.
By plugging (5.10) into (5.9) we obtain:

$$\hat{V}_{s\wedge\tau} = \hat{V}_\tau - \int_{s\wedge\tau}^{\tau} \hat{Z}_r \,\mathrm{d}W_r - \int_{s\wedge\tau}^{\tau} \left(\delta_r^{f,x+y} + \delta_r^{f,x+y}\hat{V}_r + \delta_r^{f,z}\left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)^{-1}\left(\hat{Z}_r + \hat{V}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r\right)\right) - \right.$$
$$\left. - \hat{Z}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)^{-1}\left(\hat{Z}_r + \hat{V}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r\right)\right)\right)^\top\right)\mathrm{d}r,$$

where the marked terms effectively disappear for the following reason:

Remember that $\delta_r^{\sigma,z}$, interpreted as an $d \times d$-matrix, consists of an upper left $d_1 \times d_1$ block and a lower right $d_2 \times d_2$ block. The remaining components of the matrix always vanish. This implies that the matrix $\left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)^{-1}$ will also have this structure.
Remember also that the first d_1 components of the vector $\delta^{f,z}$ vanish, which means that

$$\delta_r^{f,z}\left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)^{-1}$$

will also have this property.
Finally the last d_2 components of $\delta^{\sigma,x}$ and $\delta^{\sigma,y}$ vanish. Therefore, the expressions

$$\delta_r^{f,z}\left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)^{-1} \cdot \delta_r^{\sigma,x} \qquad \text{and} \qquad \delta_r^{f,z}\left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)^{-1} \cdot \delta_r^{\sigma,y}$$

vanish. ✓

We finally obtain

$$\hat{V}_{s\wedge\tau} = \hat{V}_\tau - \int_{s\wedge\tau}^{\tau} \hat{Z}_r \,\mathrm{d}W_r - \int_{s\wedge\tau}^{\tau} \left(\delta_r^{f,x+y} + \delta_r^{f,x+y}\hat{V}_r + \delta_r^{f,z}\left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)^{-1}\hat{Z}_r - \right.$$
$$\left. - \hat{Z}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)^{-1}\left(\hat{Z}_r + \hat{V}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r\right)\right)\right)^\top\right)\mathrm{d}r \quad (5.11)$$

a.s. for all $s \in [t_1, T]$, for almost all $x \in \mathbb{R}$. Note that this BSDE is quadratic in $\hat{Z}$.
Now, let us demonstrate that we can actually set $\tau = T$: Let

$$\tau_0 := \inf\{s \in [t_1, T] \,|\, U_s \leq 0\} \wedge T.$$

Note here that U is a continuous process starting in 1. $\hat{Z}$ is well-defined on $[t_1, \tau_0)$. Furthermore, due to (5.11) and the boundedness of $\hat{V}$ on $[t_1, T]$, $\hat{Z}$ is a BMO-process on $[t_1, \tau]$ for every stopping time $\tau < \tau_0$, with a $BMO(\mathbb{P})$ - norm which can be controlled independently of τ (Theorem A.1.11). Because of (5.10) the process $\gamma_r := \delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\tilde{Z}_r\hat{U}_r$ is also a BMO-process with the same property. This implies $\mathbb{E}\left[\int_{t_1}^{\tau_0} |\gamma_r|^2 \,\mathrm{d}r\right] < \infty$. Remember

$$U_{s\wedge\tau} = 1 + \int_{t_1}^{s\wedge\tau} \gamma_r U_r \,\mathrm{d}W_r, \quad s \in [t_1, T]$$

for all stopping times $\tau \in [t_1, \tau_0)$. This implies

$$U_\tau = \exp\left(\int_{t_1}^{\tau} \gamma_r \,\mathrm{d}W_r - \frac{1}{2}\int_{t_1}^{\tau} |\gamma_r|^2 \,\mathrm{d}r\right)$$

for all stopping times $\tau \in [t_1, \tau_0]$. Because of continuity of U this implies

$$U_{\tau_0} = \exp\left(\int_{t_1}^{\tau_0} \gamma_r \,\mathrm{d}W_r - \frac{1}{2}\int_{t_1}^{\tau_0} |\gamma_r|^2 \,\mathrm{d}r\right) > 0.$$

Now, note that $\{\tau_0 < T\} \subseteq \{U_{\tau_0} = 0\}$. However, since $U_{\tau_0} > 0$ a.s., as we have seen, $\tau_0 = T$ a.s. must hold. Therefore, $U_T > 0$ and even $U_s > 0$ for all $s \in [t_1, T]$, so we can indeed set $\tau = T$ in (5.11). ✓

Let us now rewrite (5.11) into a *linear* equation

$$\hat{V}_s = \hat{V}_T - \int_s^T \hat{Z}_r \,\mathrm{d}\tilde{W}_r - \int_s^T \left(\delta_r^{f,x+y} + \delta_r^{f,x+y}\hat{V}_r\right) \mathrm{d}r, \tag{5.12}$$

where

$$\tilde{W}_s := W_s - W_{t_1} + \int_{t_1}^s \left(\beta_r + \delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r + \delta_r^{\sigma,z}\left(I_d - \hat{V}_r\delta_r^{\sigma,z}\right)^{-1}\left(\hat{Z}_r + \hat{V}_r\left(\delta_r^{\sigma,x} + \delta_r^{\sigma,y}\hat{V}_r\right)\right)\right)^\top \mathrm{d}r,$$

for $s \in [t_1, T]$ where β is the $\mathbb{R}^{1\times d}$ - valued version of the $\mathbb{R}^{1\times(1\times d)}$- values process $\delta^{f,z}\left(I_d - \hat{V}\delta^{\sigma,z}\right)^{-1}$. $\tilde{W}$ is a Brownian motion under some probability measure $\mathbb{Q} \sim \mathbb{P}$ (Girsanov's theorem).
Let us rewrite

$$1 + \hat{V}_s = 1 + \hat{V}_T - \int_s^T \hat{Z}_r \,\mathrm{d}\tilde{W}_r - \int_s^T \delta_r^{f,x+y}\left(1 + \hat{V}_r\right) \mathrm{d}r,$$

which is a simple linear BSDE with solution

$$1 + \hat{V}_s = \mathbb{E}_{\mathbb{Q}}\left[\left(1 + \hat{V}_T\right)\exp\left(-\int_s^T \delta_r^{f,x+y}\,\mathrm{d}r\right)\middle|\mathcal{F}_s\right]$$

according to Proposition 2.2 in [EPQ97].
Remember $\delta_r^{f,x+y} \geq 0$. According to the above relationship $1 + \hat{V}_s$ can be bounded from above by $1 + \|\hat{V}_T\|_\infty \leq 1 + \left\|\frac{\mathrm{d}}{\mathrm{d}x}\xi\right\|_\infty = 1 + L_{\xi,x}$ (Lemma 2.1.4). Therefore, $\frac{\mathrm{d}}{\mathrm{d}x}u(t_1, x) = \hat{V}_{t_1} \leq L_{\xi,x} < L_{\sigma,z}^{-1}$ for a.a. $x \in \mathbb{R}$.

Secondly we have from (5.12)

$$(\hat{V}_s + L_{\sigma,z}^{-1}) = (\hat{V}_T + L_{\sigma,z}^{-1}) - \int_s^T \hat{Z}_r \,\mathrm{d}\tilde{W}_r - \int_s^T (\delta_r^{f,x+y} - \delta_r^{f,x+y}L_{\sigma,z}^{-1}) + \delta_r^{f,x+y}(\hat{V}_r + L_{\sigma,z}^{-1})\,\mathrm{d}r$$

and, therefore, using Proposition 2.2 in [EPQ97]

$$\hat{V}_{t_1} + L_{\sigma,z}^{-1} = \mathbb{E}_{\mathbb{Q}}\left[(\hat{V}_T + L_{\sigma,z}^{-1})\Gamma_T - \int_{t_1}^T \Gamma_s(\delta_s^{f,x+y} - \delta_s^{f,x+y}L_{\sigma,z}^{-1})\,\mathrm{d}s\middle|\mathcal{F}_{t_1}\right], \text{ with}$$

$$\Gamma_s = \exp\left(-\int_{t_1}^s \delta_r^{f,x+y}\,\mathrm{d}r\right).$$

Note that $\hat{V}_T + L_{\sigma,z}^{-1} \geq -\|\hat{V}_T\|_\infty + L_{\sigma,z}^{-1} > 0$ due to $L_{\xi,x} < L_{\sigma,z}^{-1}$. Also, $\Gamma_s \geq 0$ and $\delta_s^{f,x+y} - \delta_s^{f,x+y}L_{\sigma,z}^{-1} = \delta_s^{f,x+y}(1 - L_{\sigma,z}^{-1}) \leq 0$. Using simple estimates:

$$\hat{V}_{t_1} + L_{\sigma,z}^{-1} \geq (-\|\hat{V}_T\|_\infty + L_{\sigma,z}^{-1})\exp\left(-\int_{t_1}^T \|\delta_r^{f,x+y}\|_\infty\,\mathrm{d}r\right) \geq (-\|\hat{V}_T\|_\infty + L_{\sigma,z}^{-1})\exp\left(-TL_{f,x+y}\right) > 0.$$

This means that we have proven $\frac{\mathrm{d}}{\mathrm{d}x}u(t_1, x) \in [-L_{\sigma,z}^{-1} + \varepsilon, L_{\sigma,z}^{-1} - \varepsilon]$ for a.a. $x \in \mathbb{R}$, where $\varepsilon > 0$ does not depend on $t_1 \in (t_{\min}, T]$. This contradicts the statement of Lemma 5.1.12. Therefore, the assumption $I_{\max} = (t_{\min}, T]$ was wrong and $I_{\max} = [0, T]$ must hold according to Theorem 5.1.11. □

We will now apply this abstract result to a specific FBSDE, which appears in the problem of utility maximization.

Theorem 5.2.2. *Let $U : \mathbb{R} \to \mathbb{R}$ be a four times weakly differentiable function such that*

- $U'' < 0$ *everywhere,*
- $(\ln(-U''))''$ *is non-positive and bounded,*
- $\frac{U'}{U''}$ *and* $\frac{U^{(3)}}{U''}$ *are bounded.*

Let

- $\theta : \Omega \times [0,T] \to \mathbb{R}^d$ *be a progressively measurable and bounded process,*
- $H : \Omega \times \mathbb{R} \to \mathbb{R}$ *be* $\sigma\left(\mathcal{F}_0, \left(W_t + \int_0^t \pi_1(\theta_s)\,\mathrm{d}s\right)_{t\in[0,T]}\right) \otimes \mathcal{B}(\mathbb{R})$*-measurable, s.t.*
- $\|H(\cdot,0)\|_\infty < \infty$ *and* $L_{H,x} < 1$*, where* $x \in \mathbb{R}$ *refers to the second component.*

Then for all $M > 0$ *the problem given by the FBSDE*

$$X_t = x - \int_0^t \left(\pi_1(\theta_s)\frac{U'}{U''}(X_s+Y_s) + \pi_1(Z_s)\right)^\top \pi_1(\theta_s)\,\mathrm{d}s - \int_0^t \left(\pi_1(\theta_s)\frac{U'}{U''}(X_s+Y_s) + \pi_1(Z_s)\right)^\top \mathrm{d}W_s,$$

$$Y_t = H(X_T) - \int_t^T Z_s^\top\,\mathrm{d}W_s -$$

$$-\int_t^T \left(|\pi_1(\theta_s)|^2\frac{U'}{U''}\left(1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2}\right)(X_s+Y_s) + Z_s^\top\pi_1(\theta_s) - \frac{1}{2}\left(|\pi_2(Z_s)|^2 \wedge M\right)\cdot\frac{U^{(3)}}{U''}(X_s+Y_s)\right)\mathrm{d}s$$

has a unique weakly regular decoupling field $u : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$*.*
Furthermore, this FBSDE has a unique solution satisfying (5.7).

Proof. Firstly, we define a new process B, via $B_t := W_t + \int_0^t \pi_1(\theta_s)\,\mathrm{d}s$. B becomes a Brownian motion under an appropriate change of measure, such that the new measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$. Note that the filtration generated by B (and augmented by $\mathcal{F}_0$) is contained in the filtration generated by W. We will from now on work under this new measure $\mathbb{Q}$ together with the associated Brownian motion B and the corresponding filtration: Consider the FBSDE

$$X_t = x - \int_0^t \left(\pi_1(\theta_s)\frac{U'}{U''}(X_s+Y_s) + \pi_1(Z_s)\right)^\top \mathrm{d}B_s,$$

$$Y_t = H(X_T) - \int_t^T Z_s\,\mathrm{d}B_s -$$

$$-\int_t^T \left(|\pi_1(\theta_s)|^2\frac{U'}{U''}\left(1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2}\right)(X_s+Y_s) - \frac{1}{2}\left(|\pi_2(Z_s)|^2 \wedge M\right)\cdot\frac{U^{(3)}}{U''}(X_s+Y_s)\right)\mathrm{d}s. \quad (5.13)$$

It is obviously sufficient to construct a weakly regular decoupling field u for this problem instead, since the initial problem then would be solved by the same u. Using the notation of Section 5.1 the parameters μ, σ, f, ξ associated with the above problem satisfy

- $n = m = 1$,
- $\mu = 0$,

- $\sigma(t,\omega,x,y,z) := -\pi_1(\theta_t(\omega))^\top \frac{U'}{U''}(x+y) - \pi_1(z)$,
- $f(t,\omega,x,y,z) := |\pi_1(\theta_t(\omega))|^2 \frac{U'}{U''}\left(1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2}\right)(x+y) - \frac{1}{2}\left(|\pi_2(z)|^2 \wedge M\right) \cdot \frac{U^{(3)}}{U''}(x+y)$,
- $\xi = H$.

Clearly, $L_{\sigma,z} = 1$, such that $L_{H,x} < L_{\sigma,z}^{-1}$ is satisfied. Furthermore, $\sigma(t,\omega,0,0,0)$ and $f(t,\omega,0,0,0)$ are uniformly bounded. Also, σ, f are Lipschitz continuous in (x,y,z): This is due to boundedness and Lipschitz continuity of $\frac{1}{2}|\pi_2(\cdot)|^2 \wedge M$, the boundedness of $\pi_1(\theta_t(\omega))$ and the fact that the functions $\frac{U'}{U''}$, $\frac{U'}{U''}\left(1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2}\right)$, $\frac{U^{(3)}}{U''}$ are bounded and Lipschitz continuous: Note that the product or the sum of two bounded and Lipschitz continuous functions is again bounded and Lipschitz continuous. Thus we only need boundedness and Lipschitz continuity of $\frac{U'}{U''}$ and $\frac{U^{(3)}}{U''}$. Boundedness is assumed in the theorem. Furthermore

$$\left(\frac{U'}{U''}\right)' = \frac{(U'')^2 - U'U^{(3)}}{(U'')^2} = 1 - \frac{U'}{U''} \cdot \frac{U^{(3)}}{U''} \quad \text{and also} \tag{5.14}$$

$$(\ln(-U''))' = \frac{-U^{(3)}}{-U''} = \frac{U^{(3)}}{U''}, \text{ which implies}$$

$$\left(\frac{U^{(3)}}{U''}\right)' = (\ln(-U''))'', \tag{5.15}$$

where both marked expressions are bounded according to the assumptions of the theorem. ✓

So, the problem (5.13) satisfies SLC. Similarly one one can easily check that the initial FBSDE satisfies SLC as well.

With regard to other conditions of Theorem 5.2.1 we have:

- f is clearly a function of $\omega, t, x+y, \pi_2(z)$,
- $\sigma = \pi_1(\sigma)$, such that $\pi_2(\sigma)$ vanishes and σ is a function of $\omega, s, x+y, \pi_1(z)$ only,
- μ vanishes,
- $L_{\sigma,z} = 1$ as already mentioned.

It remains to verify $\frac{\mathrm{d}}{\mathrm{d}(x+y)} f \geq 0$ a.e. For this we need to check

$$\left(\frac{U'}{U''}\left(1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2}\right)\right)' \geq 0 \quad \text{and} \quad \left(\frac{U^{(3)}}{U''}\right)' \leq 0 \quad \text{a.e.}$$

The second inequality is clear, since $\left(\frac{U^{(3)}}{U''}\right)' = (\ln(-U''))'' \leq 0$ as we saw. The first requires a bit more calculation: Using the product rule together with (5.14) and (5.15):

$$\left(\frac{U'}{U''}\left(1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2}\right)\right)' = \left(1 - \frac{U'}{U''} \cdot \frac{U^{(3)}}{U''}\right)\left(1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2}\right) -$$

$$-\frac{U'}{U''}\frac{1}{2}\left((\ln(-U''))''\frac{U'}{U''} + \frac{U^{(3)}}{U''}\left(1 - \frac{U'}{U''} \cdot \frac{U^{(3)}}{U''}\right)\right) =$$

$$= \left(1 - \frac{U'}{U''} \cdot \frac{U^{(3)}}{U''}\right)\left(1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2} - \frac{U'}{U''}\frac{1}{2}\frac{U^{(3)}}{U''}\right) - \frac{U'}{U''}\frac{1}{2}(\ln(-U''))''\frac{U'}{U''} =$$

$$= \left(1 - \frac{U'}{U''} \cdot \frac{U^{(3)}}{U''}\right)^2 - \frac{1}{2}\left(\frac{U'}{U''}\right)^2 (\ln(-U''))'' \geq 0$$

Thereby Theorem 5.2.1 is applicable and we have a unique weakly regular decoupling field u on $[0,T]$ solving the problem.
The uniqueness of X, Y, Z follows from Corollary 5.1.8. $\square$

In the next result we will use processes X, Y, Z, which solve the above FBSDE, to construct optimal trading strategies for a utility maximization problem described by $U : \mathbb{R} \to \mathbb{R}$. The optimality condition is satisfied under a probability measure that differs slightly from $\mathbb{P}$, but converges to $\mathbb{P}$ for large M.
The proof is an adaptation of Theorem 3.2 in [HHI+14].

Lemma 5.2.3. *Let*

- $U : \mathbb{R} \to \mathbb{R}$ *be as in Theorem 5.2.2 and assume in addition that* $U' > 0$ *everywhere and s.t.* $\frac{U''}{U'}$ *is bounded,*
- $\theta : \Omega \times [0,T] \to \mathbb{R}^d$ *be a progressive and bounded process,*
- $H : \Omega \to \mathbb{R}$ *be* $\sigma\left(\mathcal{F}_0, \left(W_t + \int_0^t \pi_1(\theta_s)\,\mathrm{d}s\right)_{t\in[0,T]}\right)$ *- measurable and bounded.*

Define for some fixed initial value $x \in \mathbb{R}$*:*

$$\pi_s^M := -\pi_1(\theta_s)\frac{U'}{U''}(X_s^M + Y_s^M) - \pi_1(Z_s^M), \quad s \in [0,T],$$

where X^M, Y^M, Z^M *solve the FBSDE from Theorem 5.2.2 for the initial value* $x \in \mathbb{R}$ *and a given* $M > 0$ *s.t. (5.7) holds. Then*

$$\mathbb{E}_{\mathbb{P}^M}\left[U\left(x + \int_0^T (\pi_s^M)^\top(\,\mathrm{d}W_s + \theta_s\,\mathrm{d}s) + H\right)\right] = \sup_{\pi\in\mathcal{A}} \mathbb{E}_{\mathbb{P}^M}\left[U\left(x + \int_0^T \pi_s^\top(\,\mathrm{d}W_s + \theta_s\,\mathrm{d}s) + H\right)\right],$$

where

- $\mathcal{A}$ *is the set of progressive strategies* $\pi : \Omega \times [0,T] \to \mathbb{R}^d$ *s.t. the last* d_2 *components of* π *vanish and* π *is a* $BMO(\mathbb{P})$ *- process,*
- $\mathbb{P}^M \sim \mathbb{P}$, $M > 0$ *are some probability measures, such that* $\mathbb{P}^M \to \mathbb{P}$ *in the sense*

  $$\lim_{M\to\infty} \mathbb{E}_{\mathbb{P}^M}[R] = \mathbb{E}_{\mathbb{P}}[R],$$

  for all bounded random variables R.

More precisely, $\frac{\mathrm{d}\mathbb{P}^M}{\mathrm{d}\mathbb{P}} = \exp\left(\int_0^T (\zeta_s^M)^\top\,\mathrm{d}W_s - \frac{1}{2}\int_0^T |\zeta_s^M|^2\,\mathrm{d}s\right)$, *where* $\sup_{M>0} \|\zeta^M\|_{BMO(\mathbb{P})} < \infty$ *and* $\lim_{M\to\infty} \mathbb{E}\left[\left(\int_0^T |\zeta_s^M|^{p_1}\,\mathrm{d}s\right)^{p_2}\right] = 0$, *for all* $p_1 \in [1,2)$ *and* $p_2 \in [1,\infty)$.

Proof. Remember the probability measure $\mathbb{Q}$ and the process B from the proof of Theorem 5.2.2. Observing

$$Y_t^M = H - \int_t^T (Z_s^M)^\top\,\mathrm{d}B_s -$$

$$-\int_t^T \left(|\pi_1(\theta_s)|^2 \frac{U'}{U''}\left(1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2}\right)(X_s^M + Y_s^M) - \frac{1}{2}|\pi_2(Z_s^M)|^2 \wedge M \cdot \frac{U^{(3)}}{U''}(X_s^M + Y_s^M)\right)\mathrm{d}s$$

and noting that H, θ, $\frac{U'}{U''}\left(1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2}\right)$ and $\frac{U^{(3)}}{U''}$ are all uniformly bounded, we deduce that

- Y^M is bounded, according to Lemma A.1.10 applied to

$$\psi_s = |\pi_1(\theta_s)|^2 \frac{U'}{U''}\left(1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2}\right)(X_s^M + Y_s^M) - \frac{1}{2}\left(|\pi_2(Z_s^M)|^2 \wedge M\right) \cdot \frac{U^{(3)}}{U''}(X_s^M + Y_s^M)$$

and $\varphi = 0$.

- Z^M is a $BMO(\mathbb{Q}) = BMO(\mathbb{P})$ - process (Theorem A.1.11). Therefore
- Y^M is bounded by a constant independent of M according to Lemma A.1.10 applied to

$$\varphi_s = -\frac{1}{2|\pi_2(Z_s^M)|^2}\left(|\pi_2(Z_s^M)|^2 \wedge M\right) \cdot \frac{U^{(3)}}{U''}(X_s^M + Y_s^M)\pi_2(Z_s^M)^\top$$

and

$$\psi_s = |\pi_1(\theta_s)|^2 \frac{U'}{U''}\left(1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2}\right)(X_s^M + Y_s^M).$$

- Z^M is a BMO process with a $BMO(\mathbb{P})$-norm bounded by a constant independent of M (Theorem A.1.11 and Theorem A.1.6).

So, we have

$$\sup_{M>0} \|Z^M\|_{BMO(\mathbb{P})} < \infty \tag{5.16}$$

which implies in particular $\sup_{M>0} \|\pi^M\|_{BMO(\mathbb{P})} < \infty$.
By summing up the forward and the backward equations we observe that $X^M + Y^M$ has dynamics

$$\begin{aligned}
X_t^M + Y_t^M = x + Y_0^M + \int_0^t &\left(-\pi_1(\theta_s)\frac{U'}{U''}(X_s^M + Y_s^M) - \pi_1(Z_s) + Z_s^M\right)^\top \mathrm{d}W_s + \\
&+ \int_0^t \Bigg\{ \left(-\pi_1(\theta_s)\frac{U'}{U''}(X_s^M + Y_s^M) - \pi_1(Z_s^M)\right)^\top \pi_1(\theta_s)\,\mathrm{d}s + \\
+|\pi_1(\theta_s)|^2 \frac{U'}{U''}\left(1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2}\right)&(X_s^M + Y_s^M) + \left(Z_s^M\right)^\top \pi_1(\theta_s) - \frac{1}{2}\left(|\pi_2(Z_s^M)|^2 \wedge M\right) \cdot \frac{U^{(3)}}{U''}(X_s^M + Y_s^M)\Bigg\}\,\mathrm{d}s
\end{aligned}$$

a.s. for $t \in [0, T]$. Let us now apply the Itô formula to $U'(X_t^M + Y_t^M)$:

$$U'(X_t^M + Y_t^M) =$$

$$\begin{aligned}
&= U'(X_0^M + Y_0^M) + \int_0^t \Bigg\{ U''(X_s^M + Y_s^M) \Bigg\{ \left(-\pi_1(\theta_s)\frac{U'}{U''}(X_s^M + Y_s^M) - \pi_1(Z_s^M)\right)^\top \pi_1(\theta_s) + \\
&+|\pi_1(\theta_s)|^2 \frac{U'}{U''}\left(1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2}\right)(X_s^M + Y_s^M) + \pi_1(Z_s^M) \cdot \pi_1(\theta_s) - \frac{1}{2}|\pi_2(Z_s^M)|^2 \wedge M \cdot \frac{U^{(3)}}{U''}(X_s^M + Y_s^M)\Bigg\} + \\
&\quad + \frac{1}{2}U^{(3)}(X_s^M + Y_s^M)\left(|\pi_1(\theta_s)|^2 \left(\frac{U'}{U''}\right)^2 (X_s^M + Y_s^M) + |\pi_2(Z_s^M)|^2\right)\Bigg\}\,\mathrm{d}s + \\
&\quad + \int_0^t U''(X_t^M + Y_t^M)\left(-\pi_1(\theta_s)\frac{U'}{U''}(X_s^M + Y_s^M) - \pi_1(Z_s^M) + Z_s^M\right)^\top \mathrm{d}W_s,
\end{aligned}$$

where the marked terms effectively cancel out. So

$$U'(X_t^M + Y_t^M) = U'(X_0^M + Y_0^M) +$$

$$+ \int_0^t \Bigg\{ U''(X_s^M + Y_s^M) \left(\boxed{- \frac{1}{2} |\pi_1(\theta_s)|^2 \frac{U^{(3)}(U')^2}{(U'')^3}(X_s^M + Y_s^M)} - \frac{1}{2}|\pi_2(Z_s^M)|^2 \wedge M \cdot \frac{U^{(3)}}{U''}(X_s^M + Y_s^M) \right) +$$

$$+ \frac{1}{2} U^{(3)}(X_s^M + Y_s^M) \left(\boxed{|\pi_1(\theta_s)|^2 \left(\frac{U'}{U''}\right)^2 (X_s^M + Y_s^M)} + |\pi_2(Z_s^M)|^2 \right) \Bigg\} \,\mathrm{d}s +$$

$$+ \int_0^t U''(X_t^M + Y_t^M) \left(-\pi_1(\theta_s) \frac{U'}{U''}(X_s^M + Y_s^M) + \pi_2(Z_s^M) \right)^\top \mathrm{d}W_s,$$

where the marked terms effectively cancel out. So, using $a - a \wedge M = (a - M)\mathbf{1}_{a-M \geq 0}$, for $a, M \geq 0$

$$U'(X_t^M + Y_t^M) = U'(x + Y_0^M) + \int_0^t \frac{1}{2} U^{(3)}(X_s^M + Y_s^M)(|\pi_2(Z_s^M)|^2 - M)\mathbf{1}_{\{|\pi_2(Z_s^M)|^2 - M \geq 0\}} \,\mathrm{d}s +$$

$$+ \int_0^t \left(-\pi_1(\theta_s) U'(X_s^M + Y_s^M) + U''(X_t^M + Y_t^M)\pi_2(Z_s^M) \right) \mathrm{d}W_s$$

Using $\pi_1(\theta_s) \cdot \pi_2(Z_s^M) = 0$ we can rewrite this as

$$U'(X_t^M + Y_t^M) = U'(x + Y_0) + \int_0^t U'(X_s^M + Y_s^M) \left(\frac{U''}{U'}(X_s^M + Y_s^M)\pi_2(Z_s^M) - \pi_1(\theta_s) \right)^\top \cdot$$

$$\cdot \left(\mathrm{d}W_s + \frac{1}{2} \frac{U^{(3)}}{U'}(X_s^M + Y_s^M) \frac{\pi_2(Z_s^M)}{|\pi_2(Z_s^M)|^2} (|\pi_2(Z_s^M)|^2 - M)\mathbf{1}_{\{|\pi_2(Z_s^M)|^2 - M \geq 0\}} \,\mathrm{d}s \right) =$$

$$= U'(x + Y_0) + \int_0^t U'(X_s^M + Y_s^M)(\alpha_s^M)^\top \left(\mathrm{d}W_s + \zeta_s^M \,\mathrm{d}s \right), \tag{5.17}$$

where

$$\alpha_s^M := \frac{U''}{U'}(X_s^M + Y_s^M)\pi_2(Z_s^M) - \pi_1(\theta_s)$$

and

$$\zeta_s^M := \frac{1}{2} \frac{U^{(3)}}{U'}(X_s^M + Y_s^M) \frac{\pi_2(Z_s^M)}{|\pi_2(Z_s^M)|^2} (|\pi_2(Z_s^M)|^2 - M)\mathbf{1}_{\{|\pi_2(Z_s^M)|^2 - M \geq 0\}}$$

are BMO processes with $\sup_{M>0} \|\alpha^M\|_{BMO(\mathbb{P})} < \infty$ and $\sup_{M>0} \|\zeta^M\|_{BMO(\mathbb{P})} < \infty$. This follows directly from (5.16) and the boundedness of $\frac{U''}{U'}$, $\frac{U^{(3)}}{U''}$.

Now, define a Brownian motion with drift via $W_t^M := W_t + \int_0^t \zeta_s^M \,\mathrm{d}s$ and also the probability measure

$$\mathbb{P}^M := \exp\left(\int_0^T (\zeta_s^M)^\top \,\mathrm{d}W_s - \frac{1}{2} \int_0^T |\zeta_s^M|^2 \,\mathrm{d}s \right) \cdot \mathbb{P}.$$

Note Theorem A.1.2 for that. Note also that W^M is a Brownian motion under $\mathbb{P}^M$ (Girsanov's theorem). According to (5.17) the process $\frac{1}{U'(x+Y_0)} U'(X^M + Y^M)$ is a uniformly integrable exponential martingale under $\mathbb{P}^M$, since α^M is a BMO process w.r.t. $\mathbb{P}$ and, thereby, w.r.t. $\mathbb{P}^M$ (Theorem A.1.6). As a consequence of that $\mathbb{E}_{\mathbb{P}^M}[(U'(X_T^M + Y_T^M))^q] < \infty$ for some $q > 1$ (Theorem A.1.9).

Now, take any $h \in \mathcal{A}$. Since $\pi_2(h) = 0$ we have $h^\top \alpha^M = -h^\top \pi_1(\theta) = -h^\top \theta$ and $h^\top \zeta^M = 0$. Now, consider

$$U'(X_T^M + Y_T^M) \int_0^T h_s^\top (\,\mathrm{d}W_s + \theta_s \,\mathrm{d}s) = U'(X_T^M + Y_T^M) \int_0^T h_s^\top (\,\mathrm{d}W_s^M + \theta_s \,\mathrm{d}s).$$

Applying Itô's formula and using (5.17):

$$U'(X_t^M + Y_t^M) \int_0^t h_s^\top (\,\mathrm{d}W_s^M + \theta_s \,\mathrm{d}s) = \int_0^t \Big(U'(X_s^M + Y_s^M) h_s^\top \theta_s + U'(X_s^M + Y_s^M)(\alpha_s^M)^\top h_s \Big) \,\mathrm{d}s +$$

$$+ \int_0^t \left(U'(X_s^M + Y_s^M) h_s + U'(X_s^M + Y_s^M) \alpha_s^M \int_0^s h_r^\top (\,\mathrm{d}W_r^M + \theta_r \,\mathrm{d}r) \right)^\top \,\mathrm{d}W_s^M.$$

Using the definition of α^M we see that the final variation part on the right-hand side disappears and

$$U'(X^M + Y^M) \int_0^\cdot h_s^\top (\,\mathrm{d}W_s^M + \theta_s \,\mathrm{d}s)$$

is a local martingale under $\mathbb{P}^M$. It is also uniformly integrable: For any $q' \in (1, q)$ and the associated $p' = \frac{q'}{q'-1} > 1$ we have using Hölder and BDG inequalities:

$$\mathbb{E}_{\mathbb{P}^M} \left[\left(U'(X_t^M + Y_t^M) \int_0^t h_s^\top (\,\mathrm{d}W_s^M + \theta_s \,\mathrm{d}s) \right)^{\frac{q}{q'}} \right] \leq$$

$$\leq \Big(\mathbb{E}_{\mathbb{P}^M} \left[(U'(X_t^M + Y_t^M))^q \right] \Big)^{\frac{1}{q'}} \left(\mathbb{E}_{\mathbb{P}^M} \left[\left(\int_0^t h_s^\top (\,\mathrm{d}W_s^M + \theta_s \,\mathrm{d}s) \right)^{\frac{qp'}{q'}} \right] \right)^{\frac{1}{p'}} \leq$$

$$\leq \Big(\mathbb{E}_{\mathbb{P}^M} \left[(U'(X_t^M + Y_t^M))^q \right] \Big)^{\frac{1}{q'}} \left(\mathbb{E}_{\mathbb{P}^{M,\theta}} \left[\frac{\mathrm{d}\mathbb{P}^M}{\mathrm{d}\mathbb{P}^{M,\theta}} \left(\int_0^t h_s^\top \,\mathrm{d}W_s^{M,\theta} \right)^{\frac{qp'}{q'}} \right] \right)^{\frac{1}{p'}} \leq$$

$$\leq \Big(\mathbb{E}_{\mathbb{P}^M} \left[(U'(X_t^M + Y_t^M))^q \right] \Big)^{\frac{1}{q'}} \left(\mathbb{E}_{\mathbb{P}^{M,\theta}} \left[\left(\frac{\mathrm{d}\mathbb{P}^M}{\mathrm{d}\mathbb{P}^{M,\theta}} \right)^2 \right] \mathbb{E}_{\mathbb{P}^{M,\theta}} \left[\left(\int_0^t h_s^\top \,\mathrm{d}W_s^{M,\theta} \right)^{\frac{2qp'}{q'}} \right] \right)^{\frac{1}{2p'}} \leq$$

$$\leq \Big(\mathbb{E}_{\mathbb{P}^M} \left[(U'(X_T^M + Y_T^M))^q \right] \Big)^{\frac{1}{q'}} \left(C \cdot \mathbb{E}_{\mathbb{P}^{M,\theta}} \left[\left(\int_0^T |h_s|^2 \,\mathrm{d}s \right)^{\frac{qp'}{q'}} \right] \right)^{\frac{1}{2p'}} < \infty,$$

where $\mathbb{P}^{M,\theta}$ is some measure turning $W^{M,\theta} := W^M + \int_0^\cdot \theta_s \,\mathrm{d}s$ into a Brownian motion and having a BMO process as exponential density in $\frac{\mathrm{d}\mathbb{P}^{M,\theta}}{\mathrm{d}\mathbb{P}}$, s.t. h is still a BMO process w.r.t. $\mathbb{P}^{M,\theta}$ (Theorem A.1.6) and we can apply Lemma A.1.3.
Thus, we have shown

$$\mathbb{E}_{\mathbb{P}^M} \left[U'(X_T^M + Y_T^M) \int_0^T h_s^\top (\,\mathrm{d}W_s + \theta_s \,\mathrm{d}s) \right] = 0. \tag{5.18}$$

Now, choose any $\pi \in \mathcal{A}$. Then due to concavity of U:

$$U \left(x + \int_0^T (\pi_s)^\top (\,\mathrm{d}W_s + \theta_s \,\mathrm{d}s) + H \right) \leq U \left(x + \int_0^T (\pi_s^M)^\top (\,\mathrm{d}W_s + \theta_s \,\mathrm{d}s) + H \right) +$$

$$+ U' \left(x + \int_0^T (\pi_s^M)^\top (\,\mathrm{d}W_s + \theta_s \,\mathrm{d}s) + H \right) \int_0^T (\pi_s - \pi_s^M)^\top (\,\mathrm{d}W_s + \theta_s \,\mathrm{d}s)$$

Note $x + \int_0^T (\pi_s^M)^\top (\,\mathrm{d}W_s + \theta_s \,\mathrm{d}s) + H = X_T^M + Y_T^M$ according to the definition of π^M and the forward equation. Also, $\pi - \pi^M \in \mathcal{A}$. Applying expectations on both sides of the above inequality an using (5.18):

$$\mathbb{E}_{\mathbb{P}^M} \left[U \left(x + \int_0^T (\pi_s)^\top (\,\mathrm{d}W_s + \theta_s \,\mathrm{d}s) + H \right) \right] \leq \mathbb{E}_{\mathbb{P}^M} \left[U \left(x + \int_0^T (\pi_s^M)^\top (\,\mathrm{d}W_s + \theta_s \,\mathrm{d}s) + H \right) \right].$$

This already shows the optimality of π^M.
Now, we would like to show convergence of ζ^M: Note that for all $z \in \mathbb{R}$ and all $p_1 \in [1, 2)$:

$$\left(\frac{z^2}{M}\right)^{\frac{p_1}{2}} \mathbf{1}_{\{z^2 \geq M\}} \leq \frac{z^2}{M} \mathbf{1}_{\{z^2 \geq M\}}.$$

This implies $z^{p_1} \mathbf{1}_{\{z^2 \geq M\}} \leq \frac{1}{M^{1-\frac{p_1}{2}}} z^2 \mathbf{1}_{\{z^2 \geq M\}} \leq \frac{1}{M^{1-\frac{p_1}{2}}} z^2$. Now, observe from the definition of ζ^M:

$$|\zeta_s^M|^{p_1} \leq C_1 \frac{1}{|\pi_2(Z_s^M)|^{p_1}} (|\pi_2(Z_s^M)|^2 - M)^{p_1} \mathbf{1}_{\{|\pi_2(Z_s^M)|^2 \geq M\}} \leq C_1 |\pi_2(Z_s^M)|^{p_1} \mathbf{1}_{\{|\pi_2(Z_s^M)|^2 \geq M\}} \leq$$

$$\leq \frac{C_1}{M^{1-\frac{p_1}{2}}} |\pi_2(Z_s^M)|^2 \leq \frac{C_1}{M^{1-\frac{p_1}{2}}} |Z_s^M|^2,$$

where $C_1 \in [0, \infty)$ is some constant determined by U alone. Therefore, for all $p_2 \in [1, \infty)$:

$$\mathbb{E}\left[\left(\int_0^T |\zeta_s^M|^{p_1} \, \mathrm{d}s\right)^{p_2}\right] \leq \left(\frac{C_1}{M^{1-\frac{p_1}{2}}}\right)^{p_2} \mathbb{E}\left[\left(\int_0^T |Z_s^M|^2 \, \mathrm{d}s\right)^{p_2}\right] \leq$$

$$\leq \left(\frac{C_1}{M^{1-\frac{p_1}{2}}}\right)^{p_2} \left(C_2 \sup_{M>0} \|Z^M\|_{BMO(\mathbb{P})}\right)^{p_2} \to 0 \text{ as } M \to \infty,$$

where we used Lemma A.1.3. $C_2 \in [0, \infty)$ is some constant determined by p_2 alone.
Convergence of $\mathbb{P}^M$ to $\mathbb{P}$ now follows directly from Lemma A.4.3: Condition (A.6) in this lemma is satisfied due to $\sup_{M>0} \|\zeta^M\|_{BMO(\mathbb{P})} < \infty$ and Theorem A.1.9. □

Remark 5.2.4. We can define a major class of utility functions satisfying the conditions of the preceding theorem: Let $\kappa : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that

- κ is twice weakly differentiable with bounded first and second derivatives,
- $\kappa' \geq \varepsilon > 0$,
- $\kappa'' \geq 0$ a.e..

A linear strictly increasing function would probably be the simplest function κ fulfilling these requirements. Note that κ' is non-decreasing and bounded, therefore $\lim_{x\to\infty} \kappa'(x)$ and $\lim_{x\to-\infty} \kappa'(x)$ exist. Also, $\lim_{x\to-\infty} \kappa'(x) \geq \varepsilon > 0$. For very large or very small values κ behaves linearly. It is also convex and strictly increasing.
Now, define

$$U''(x) := -\exp(-\kappa(x)),$$

$$U'(x) := \int_x^\infty \exp(-\kappa(y)) \, \mathrm{d}y,$$

$$U(x) := -\int_x^\infty \int_y^\infty \exp(-\kappa(z)) \, \mathrm{d}z \, \mathrm{d}y.$$

Note that $\exp(-\kappa(x)) \leq \exp(-\gamma x)$ for some fixed $\gamma > 0$ and sufficiently large x. Therefore, the expression $\int_x^\infty \exp(-\kappa(y)) \, \mathrm{d}y$ is well-defined for all $x \in \mathbb{R}$ and bounded by $\frac{1}{\gamma} \exp(-\gamma x)$ for sufficiently large x. Therefore, the expression

$$\int_x^\infty \int_y^\infty \exp(-\kappa(z)) \, \mathrm{d}z \, \mathrm{d}y$$

is also well-defined. We observe furthermore:

- $U'(x) > 0$ for all $x \in \mathbb{R}$,
- $U''(x) < 0$ for all $x \in \mathbb{R}$,
- $(\ln(-U''))'' = -\kappa''(x)$ is non-positive and bounded.

We claim that $\frac{U'}{U''}$ is bounded:

Proof. Clearly, $\frac{U'}{U''} < 0$. Also, $\frac{U'}{U''}$ is continuous. Let us analyze its behaviour for $x \to \infty$ and $x \to -\infty$. We do this by L'Hôpital's rule: Clearly, $\lim_{x\to\infty} U''(x) = 0$, $\lim_{x\to\infty} U'(x) = 0$, $\lim_{x\to-\infty} U''(x) = -\infty$, $\lim_{x\to-\infty} U'(x) = \infty$, due to $\inf_{x\in\mathbb{R}} \kappa'(x) > 0$. We have

$$\lim_{x\to\infty} \frac{U'(x)}{U''(x)} = \lim_{x\to\infty} \frac{-\exp(-\kappa(x))}{\kappa'(x)\exp(-\kappa(x))} = \lim_{x\to\infty} \frac{-1}{\kappa'(x)} = -\frac{1}{\lim_{x\to\infty} \kappa'(x)} > -\infty$$

and similarly

$$\lim_{x\to-\infty} \frac{U'(x)}{U''(x)} = -\frac{1}{\lim_{x\to-\infty} \kappa'(x)} > -\infty.$$

Since $\frac{U'}{U''}$ is continuous $\inf_{x\in\mathbb{R}} \frac{U'}{U''}(x) > -\infty$ must hold and so $\frac{U'}{U''}$ is bounded. □

We easily see that $\frac{U^{(3)}}{U''}$ is also bounded: $\frac{U^{(3)}}{U''} = \frac{\kappa' \exp(-\kappa)}{-\exp(-\kappa)} = -\kappa'$ which is bounded.

Finally we have to demonstrate that $\frac{U''}{U'}$ is bounded as well:

Proof. This is again done by taking into account that $\frac{U''}{U'}$ is negative and continuous. Using L'Hôpital's rule:

$$\lim_{x\to\infty} \frac{U''(x)}{U'(x)} = \lim_{x\to\infty} \frac{\kappa'(x)\exp(-\kappa(x))}{-\exp(-\kappa(x))} = -\lim_{x\to\infty} \kappa'(x) > -\infty,$$
$$\lim_{x\to-\infty} \frac{U''(x)}{U'(x)} = \lim_{x\to-\infty} \frac{\kappa'(x)\exp(-\kappa(x))}{-\exp(-\kappa(x))} = -\lim_{x\to-\infty} \kappa'(x) > -\infty.$$

□

We have, thus, concluded that U is a utility function satisfying the conditions of Lemma 5.2.3.

In the terminology of exponential utility functions, which are a special case of utilities U considered above, $\kappa'(x)$ can be interpreted as the "local risk aversion" which unlike in the case of exponential utility is allowed to change with x. We essentially require that κ' is strictly positive, which is self evident, but also, that it is increasing. The latter means, that if our position x is small, we will trade aggressively with low risk aversion, because we have nothing to lose, but if our position is large, for instance due to good profits in the past, we will prefer more conservative trading strategies to lock in the gains.

The fact that we require U'' to have the structure $-\exp(-\kappa(x))$ with some sufficiently smooth function κ is not really restrictive due to the fact that it already follows from assuming that U is strictly concave (it already has to be concave in order to be a utility function) and sufficiently smooth. Neither do we consider differentiability and boundedness assumptions for various functions associated with U structurally restrictive. The only "hard" restrictions are $\kappa' \geq \varepsilon > 0$ and $\kappa'' \geq 0$ both of which make sense in general as we have motivated above.

Finally note that from the assumption of U'' having the form $-\exp(-\kappa(x))$ with a regular κ as required in the beginning we immediately obtain that U itself can be written in the form

$$U(x) = C_1 + C_2 x - \int_x^\infty \int_y^\infty \exp(-\kappa(z))\,\mathrm{d}z\,\mathrm{d}y,$$

with some constants $C_1, C_2 \in \mathbb{R}$. Now, C_2 vanishes as a consequence of $\frac{U''}{U'}$ and $\frac{U'}{U''}$ being bounded, while C_1 can be set to 0 since additive constants do not change the nature of the maximization problem.

5.3 Main results for the Markovian Case

The main goal of this section is to prove Theorem 5.3.13, which basically states that FBSDE (5.4) has a unique solution under certain assumptions on U, θ and H:

- U belongs to the class of utility functions described in Remark 5.2.4 while
- θ and H depend on ω through a standard, possibly high dimensional, diffusion.

This particular structure of U has been motivated in Remark 5.2.4 already. The second assumption might be motivated by the following heuristic arguments:

- We suspect that it is possible to adequately approximate every "non-pathological" H by an H with structural properties required for Theorem 5.3.13: First approximate H by a deterministic function of $(W_{t_i})_{i=1,\ldots,N}$ for finitely many times t_i and then approximate every W_{t_i} by the terminal value of a standard forward diffusion with vanishing drift and a volatility which assumes values between 0 and 1, is close to 1 before time t_i and close to 0 after that, such that W_{t_i} is isolated.
- A similar approximative argument could be applied to θ.
- In general, when trying to treat FBSDEs numerically assumptions which make the problem Markovian are usually made anyway (e.g. [BD07]).

We will make use of an extension of the theory of decoupling fields for the Markovian case, i.e. the case where μ, σ, f, ξ do not depend on ω, in which the assumption of Lipschitz continuity of these parameter functions can be partially dropped. This will allow us to remove the cutoff employed in the previous section for the purpose of ensuring Lipschitz continuity. So, the trading strategy obtained from the solution of this truly quadratic FBSDE will be optimal under the initial measure $\mathbb{P}$ and not some measure $\mathbb{P}^M$ close to it (as was the case in the previous section).
As a by-product of the use of the Markovian theory of decoupling fields we will obtain boundedness of Z and, thereby, boundedness of the optimal trading strategy.
The string of arguments is similar to the one employed in Theorem 5.2.1: We control the spatial derivative of the decoupling field by more or less explicitly deducing its dynamics. However, the calculations become more complex since the forward part of the problem is high-dimensional and consists of two different parts with different roles. Also, the non-boundedness of $\frac{\mathrm{d}}{\mathrm{d}(x+y)}f$, where f is the generator in the backward equation, leads to problems which have to be overcome through a deeper exploitation of the particular structure of the FBSDE than was necessary in Theorem 5.2.1.

In the following subsection we will briefly summarize the key results of the abstract theory of Markovian decoupling fields, we will rely on later in the section. The presented theory is derived from the SLC theory of Chapter 2 and proven in Chapter 4.

5.3.1 Decoupling fields for Markovian problems

A problem given by μ, σ, f, ξ is said to be *Markovian*, if these four functions are deterministic, i.e. depend on t, x, y, z only. In the Markovian case we can somewhat relax the Lipschitz continuity assumption and still obtain local existence together with uniqueness. What makes the Markovian case so special is the property

$$"Z_s = u_x(s, X_s) \cdot \sigma(s, X_s, Y_s, Z_s)"$$

which comes from the fact that u will also be deterministic. This property allows us to bound Z by a constant if we assume that σ is bounded.

This potential boundedness of Z in the Markovian case motivates the following definition, which will allow us to develop a theory for non-Lipschitz problems:

Definition 5.3.1. Let $\xi : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ be measurable and let $t \in [0, T]$.
We call a function $u : [t, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ with $u(T, \omega, \cdot) = \xi(\omega, \cdot)$ for a.a. $\omega \in \Omega$ a *Markovian decoupling field* for $(\xi, (\mu, \sigma, f))$ on $[t, T]$ if for all $t_1, t_2 \in [t, T]$ with $t_1 \leq t_2$ and any $\mathcal{F}_{t_1}$ - measurable $X_{t_1} : \Omega \to \mathbb{R}^n$ there exist progressive processes X, Y, Z on $[t_1, t_2]$ such that

- $X_s = X_{t_1} + \int_{t_1}^s \mu(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_{t_1}^s \sigma(r, X_r, Y_r, Z_r)\,\mathrm{d}W_r$ a.s.,
- $Y_s = Y_{t_2} - \int_s^{t_2} f(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^{t_2} Z_r\,\mathrm{d}W_r$ a.s.,
- $Y_s = u(s, X_s)$ a.s.

for all $s \in [t_1, t_2]$ and such that $\|Z\|_\infty < \infty$ holds.
In particular, we want all integrals to be well-defined and X, Y, Z to have values in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^{m \times d}$ respectively.
Furthermore, we call a function $u : (t, T] \times \mathbb{R}^n \to \mathbb{R}^m$ a Markovian decoupling field for $(\xi, (\mu, \sigma, f))$ on $(t, T]$ if u restricted to $[t', T]$ is a Markovian decoupling field for all $t' \in (t, T]$.

Note that a Markovian decoupling field is always a decoupling field in the standard sense as well. The only difference is that we are only interested in X, Y, Z, where Z is a.s. bounded.
Regularity for Markovian decoupling fields is defined very similarly to standard regularity:

Definition 5.3.2. Let $u : [t, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ be a Markovian decoupling field to $(\xi, (\mu, \sigma, f))$. We call u *weakly regular*, if $L_{u,x} < L_{\sigma,z}^{-1}$ and $\sup_{s \in [t,T]} \|u(s, \cdot, 0)\|_\infty < \infty$.

Furthermore, we call a weakly regular u *strongly regular* if for all fixed $t_1, t_2 \in [t, T]$, $t_1 \leq t_2$, the processes X, Y, Z arising in the defining property of a Markovian decoupling field are a.e. unique for each *constant* initial value $X_{t_1} = x \in \mathbb{R}^n$ and satisfy (5.5). In addition they must be measurable as functions of (x, s, ω) and even weakly differentiable w.r.t. $x \in \mathbb{R}^n$ such that for every $s \in [t_1, t_2]$ the mappings X_s and Y_s are measurable functions of (x, ω) and even weakly differentiable w.r.t. x such that (5.6) holds.

We say that a Markovian decoupling field u on $[t, T]$ is *strongly regular* on a subinterval $[t_1, t_2] \subseteq [t, T]$ if u restricted to $[t_1, t_2]$ is a strongly regular Markovian decoupling field for $(u(t_2, \cdot), (\mu, \sigma, f))$.
Furthermore, we say that a Markovian decoupling field $u : (t, T] \times \mathbb{R}^n \to \mathbb{R}^m$

- is weakly regular if u restricted to $[t', T]$ is weakly regular for all $t' \in (t, T]$,
- is strongly regular if u restricted to $[t', T]$ is strongly regular for all $t' \in (t, T]$.

Now, we define a class of problems, for which an existence and uniqueness theory will be developed:

Definition 5.3.3. We say that ξ, μ, σ, f satisfy *modified local Lipschitz conditions (MLLC)* if

- μ, σ, f are
  - deterministic,
  - Lipschitz continuous in x, y, z on sets of the form $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times B$, where $B \subset \mathbb{R}^{m \times d}$ is an arbitrary bounded set

- and such that $\|\mu(\cdot, 0, 0, 0)\|_\infty, \|f(\cdot, 0, 0, 0)\|_\infty, \|\sigma(\cdot, \cdot, \cdot, 0)\|_\infty, L_{\sigma,z} < \infty$,

- $\xi : \mathbb{R}^n \to \mathbb{R}^m$ satisfies $L_{\xi,x} < L_{\sigma,z}^{-1}$,

where $L_{\sigma,z}$ denotes the Lipschitz constant of σ w.r.t. the dependence on the last component z (and w.r.t. the Frobenius norms on $\mathbb{R}^{m\times d}$ and $\mathbb{R}^{n\times d}$). By $L_{\sigma,z}^{-1} = \frac{1}{L_{\sigma,z}}$ we mean $\frac{1}{L_{\sigma,z}}$ if $L_{\sigma,z} > 0$ and ∞ otherwise.

The following natural concept introduces a type of Markovian decoupling field for non-Lipschitz problems (non-Lipschitz in z), to which nevertheless standard Lipschitz results can be applied.

Definition 5.3.4. Let u be a Markovian decoupling field for $(\xi, (\mu, \sigma, f))$. We call u *controlled in z* if there exists a constant $C > 0$ such that for all $t_1, t_2 \in [t, T]$, $t_1 \le t_2$, and all initial values X_{t_1}, the corresponding processes X, Y, Z from the definition of a Markovian decoupling field satisfy $|Z_s(\omega)| \le C$, for almost all $(s, \omega) \in [t, T] \times \Omega$. If for a fixed triple (t_1, t_2, X_{t_1}) there are different choices for X, Y, Z, then all of them are supposed to satisfy the above control.

We say that a Markovian decoupling field u on $[t, T]$ is *controlled in z* on a subinterval $[t_1, t_2] \subseteq [t, T]$ if u restricted to $[t_1, t_2]$ is a Markovian decoupling field for $(u(t_2, \cdot), (\mu, \sigma, f))$ that is controlled in z.

Furthermore, we call a Markovian decoupling field on an interval $(s, T]$ *controlled in z* if it is controlled in z on every compact subinterval $[t, T] \subseteq (s, T]$ (with C possibly depending on t).

The following important result allows us to connect the MLLC - case to SLC.

Theorem 5.3.5 (Theorem 4.2.23 in Chapter 4.). *Let μ, σ, f, ξ satisfy MLLC and assume that there exists a weakly regular Markovian decoupling field u to this problem on some interval $[t, T]$. Then u is controlled in z.*

Note at this point that such a u will be a standard decoupling field to an SLC problem if we cutoff μ, σ, f appropriately. We can, thereby, extend the whole SLC theory to MLLC problems:

Theorem 5.3.6 (Global uniqueness). *Let μ, σ, f, ξ satisfy MLLC and assume that there are two weakly regular Markovian decoupling fields $u^{(1)}, u^{(2)}$ to this problem on some interval $[t, T]$. Then $u^{(1)} = u^{(2)}$ (up to modifications).*

Proof. Theorem 4.2.24 in Chapter 4. □

Theorem 5.3.7 (Global regularity). *Let μ, σ, f, ξ satisfy MLLC and assume that there exists a weakly regular Markovian decoupling field u to this problem on some interval $[t, T]$. Then u is strongly regular.*

Proof. Theorem 4.2.24 in Chapter 4. □

Lemma 5.3.8 (Lemma 4.2.25 in Chapter 4.). *Let μ, σ, f, ξ satisfy MLLC and assume that there exists a weakly regular Markovian decoupling field u on some interval $[t, T]$.*
Then for any initial condition $X_t = x \in \mathbb{R}^n$ there is a unique solution (X, Y, Z) of the FBSDE on $[t, T]$ s.t.

$$\sup_{s\in[t,T]} \mathbb{E}_{0,\infty}[|X_s|^2] + \sup_{s\in[t,T]} \mathbb{E}_{0,\infty}[|Y_s|^2] + \|Z\|_\infty < \infty.$$

Definition 5.3.9. Let $I_{\max}^M \subseteq [0, T]$ for $(\xi, (\mu, \sigma, f))$ be the union of all intervals $[t, T] \subseteq [0, T]$ such that there exists a weakly regular Markovian decoupling field u on $[t, T]$.

Theorem 5.3.10 (Global existence in weak form). *Let μ, σ, f, ξ satisfy MLLC. Then there exists a unique weakly regular Markovian decoupling field u on $I_{\max}^M$. This u is also controlled in z, strongly regular, deterministic and continuous.*
Furthermore, either $I_{\max}^M = [0, T]$ or $I_{\max}^M = (t_{\min}^M, T]$, where $0 \le t_{\min}^M < T$.

Proof. Theorem 4.2.28 in Chapter 4. □

The following result basically states that for a singularity $t_{\min}^M$ to occur u_x has to "explode" at $t_{\min}^M$.

Lemma 5.3.11 (Lemma 4.2.29 in Chapter 4)**.** *Let μ, σ, f, ξ satisfy MLLC. If $I_{\max}^M = (t_{\min}^M, T]$, then*

$$\lim_{t \downarrow t_{\min}^M} L_{u(t,\cdot),x} = L_{\sigma,z}^{-1},$$

where u is the unique weakly regular Markovian decoupling field from Theorem 5.3.10.

5.3.2 Solving the FBSDE - an abstract result

For some $\varepsilon > 0$ consider a forward backward system of the form

$$\begin{aligned}
\tilde{X}_s = \tilde{x} + \int_t^s \frac{1}{\varepsilon}\tilde{\mu}(r, \varepsilon\tilde{X}_r)\,\mathrm{d}r + \int_t^s \mathrm{d}W_r^\top \frac{1}{\varepsilon}\tilde{\sigma}(r, \varepsilon\tilde{X}_r),& \\
\overline{X}_s = \overline{x} + \int_t^s \overline{\mu}(r, \varepsilon\tilde{X}_r)\,\mathrm{d}r + \int_t^s \mathrm{d}W_r^\top \overline{\sigma}(r, \varepsilon\tilde{X}_r, \overline{X}_r, Y_r, Z_r),& \\
Y_s = \xi(\varepsilon\tilde{X}_T, \overline{X}_T) - \int_s^T f(r, \varepsilon\tilde{X}_r, \overline{X}_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^T \mathrm{d}W_r^\top Z_r&
\end{aligned} \tag{5.19}$$

a.s. for all $s \in [t, T]$, where $\tilde{X}$ is N-dimensional, $N \in \mathbb{N}$, and $\overline{X}, Y$ are real-valued. We assume that

- $\tilde{\mu}, \tilde{\sigma}, \overline{\mu} : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N, \mathbb{R}^{d\times N}, \mathbb{R}$ are measurable and Lipschitz continuous in the second component with Liptschitz constants $L_{\tilde{\mu},\tilde{x}}$, $L_{\tilde{\sigma},\tilde{x}}$, $L_{\overline{\mu},\tilde{x}}$ and such that $\|\tilde{\mu}(\cdot,0)\|_\infty, \|\tilde{\sigma}\|_\infty, \|\overline{\mu}(\cdot,0)\|_\infty < \infty$,
- $\overline{\sigma} : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is measurable, Lipschitz continuous in the last four components and satisfies $\|\overline{\sigma}(\cdot,\cdot,\cdot,\cdot,0)\|_\infty < \infty$,
- $f : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is measurable and Lipschitz continuous in the last four components on sets of the form $[0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \times B$, where $B \subseteq \mathbb{R}^d$ is bounded. We also assume $\|f(\cdot,0,0,0,0)\|_\infty < \infty$.
- $\xi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous in both components with the two Lipschitz constants $L_{\xi,\tilde{x}}$ and $L_{\xi,\overline{x}}$. We assume that $L_{\xi,\overline{x}} < L_{\overline{\sigma},z}^{-1}$, where $L_{\overline{\sigma},z}$ refers to the Lipschitz constant of $\overline{\sigma}$ w.r.t. the last component. Furthermore, L_ξ refers to the Lipschitz constant of ξ w.r.t. the Euclidian norm on $\mathbb{R}^N \times \mathbb{R}$.

The problem is to find progressively measurable processes $\tilde{X}, \overline{X}, Y, Z$ s.t.

- $\tilde{X}$ is $\mathbb{R}^N$ - valued,
- $\overline{X}$ and Y are $\mathbb{R}$ - valued,
- Z is $\mathbb{R}^d$ - valued

and such that (5.19) is satisfied.

Note that for varying $\varepsilon > 0$ the different problems are equivalent to each other in the following sense: If $\tilde{X}^{\varepsilon_1}, \overline{X}, Y, Z$ solve (5.19) for some $\varepsilon_1 > 0$ on some interval $[t,T]$, then $\tilde{X}^{\varepsilon_2} := \frac{\varepsilon_1}{\varepsilon_2}\tilde{X}^{\varepsilon_1}, \overline{X}, Y, Z$ solve (5.19) for some $\varepsilon_2 > 0$ on the same interval. This means that we can choose the parameter $\varepsilon > 0$ as we like without changing the nature of the problem. In particular, if we define the terminal condition ξ^ε via $\xi^\varepsilon(\tilde{x},\overline{x}) := \xi(\varepsilon\tilde{x},\overline{x})$, we can ensure that the Lipschitz constant L_{ξ^ε} of ξ^ε satisfies $L_{\xi^\varepsilon} < L_{\overline{\sigma},z}$ by choosing ε small enough! This explains why we work with the parameter $\varepsilon > 0$.

Also, note that (5.19) describes a Markovian problem, which satisfies MLLC (for ε small enough), such that the theory previously described is well applicable: The forward equation is $N+1$ - dimensional and the backward equation has dimension 1. Also, observe that the first N components of the forward equation do not depend on the rest of the problem, i.e. $\tilde{X}$ depends only on the parameters $\tilde{x}, \tilde{\mu}, \tilde{\sigma}$ and ε.

Again we assume that we have $d_1 \in \mathbb{N}$, $d_2 \in \mathbb{N}_0$ such that $d_1 + d_2 = d$, which is the dimension of our Brownian motion W.

We make the following structural requirements for f:

- f can be written as a function of $t, \tilde{x}, \overline{x}+y, \pi_2(z)$,
- f is (classically) differentiable in $(\tilde{x}, \overline{x}+y, z)$ everywhere with $\frac{\mathrm{d}}{\mathrm{d}(\overline{x}+y)} f \geq 0$,
- $|\frac{\mathrm{d}}{\mathrm{d}z} f(s,\tilde{x},\overline{x}+y,z)| \leq C(1+|z|)$ for all $s,\tilde{x},\overline{x},y,z$ with some constant $C > 0$,
- $|\frac{\mathrm{d}}{\mathrm{d}(\overline{x}+y)} f(s,\tilde{x},\overline{x}+y,z)| \leq C(1+|z|^2)$ for all $s,\tilde{x},\overline{x},y,z$ with some constant $C > 0$,
- $\|f(\cdot,\cdot,\cdot,\cdot,0)\|_\infty < \infty$,
- $\left\|\frac{\mathrm{d}}{\mathrm{d}\tilde{x}} f\right\|_\infty < \infty$.

We make the following structural assumptions for $\overline{\sigma}$ and ξ:

- $\overline{\sigma}$ has the form $\overline{\sigma} = \begin{pmatrix} \overline{\sigma}^{(1)} \\ \overline{\sigma}^{(2)} \end{pmatrix}$, with a d_1 - dimensional $\overline{\sigma}^{(1)}$ and d_2 - dimensional $\overline{\sigma}^{(2)}$, such that $\overline{\sigma}^{(1)}$ is a function of $t,\tilde{x},\overline{x},y,\pi_1(z)$ and $\overline{\sigma}^{(2)}$ is a function of $t,\tilde{x},\pi_2(z)$,
- $\overline{\sigma}$ is differentiable in $(\tilde{x},\overline{x},y,z)$ everywhere with bounded derivatives,
- $L_{\xi,\overline{x}} < 1 \leq L_{\overline{\sigma},z}^{-1}$,
- $\|\xi\|_\infty < \infty$.

Under these conditions we can prove the following abstract result, which will be applied to the particular FBSDE later on. It basically states that for $I^M_{\max} = [0,T]$ to hold it is enough to control the Lipschitz constant of u w.r.t. $\overline{x} \in \mathbb{R}$: The Lipschitz constant w.r.t. $\tilde{x} \in \mathbb{R}^N$ will then be controlled automatically as well.

Theorem 5.3.12. *Assume that the above problem has the following property: Every weakly regular Markovian decoupling field $u : [t,T] \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies*

$$\sup_{s\in[t,T]} \left\| \frac{\mathrm{d}}{\mathrm{d}\overline{x}} u(s,\cdot,\cdot) \right\|_\infty \leq K$$

for all t, u, $\varepsilon \in (0,\varepsilon_0]$, where $K < L_{\overline{\sigma},z}^{-1}$ and $\varepsilon_0 > 0$.
Then there exists an $\varepsilon > 0$ such that for the above problem $I^M_{\max} = [0,T]$ holds true.

Proof. Assume $I^M_{\max} = (t^M_{\min}, T]$ with some $t^M_{\min} \in [0,T)$. Let from now on u be the weakly regular Markovian decoupling field from Theorem 5.3.10 defined on the whole of $I^M_{\max}$. We assume without loss of generality that u is a function on $I^M_{\max} \times \mathbb{R}^{N+1} \times \mathbb{R}$. u is controlled in z so it is also a decoupling field to an SLC problem, which has all the properties of the MLLC problem we consider. Thereby, we can adopt calculations performed in the proof of Lemma 5.2.1. Note also that u is strongly regular.

Choose any $t_1 \in (t^M_{\min}, T]$, any $\tilde{x} \in \mathbb{R}^N, \overline{x} \in \mathbb{R}$ and consider the corresponding FBSDE on $[t_1,T]$:

- $\tilde{X}_s = \tilde{x} + \int_{t_1}^s \frac{1}{\varepsilon}\tilde{\mu}(r, \varepsilon\tilde{X}_r)\,\mathrm{d}r + \int_{t_1}^s \mathrm{d}W_r^\top \frac{1}{\varepsilon}\tilde{\sigma}(r, \varepsilon\tilde{X}_r)$,
- $\overline{X}_s = \overline{x} + \int_{t_1}^s \overline{\mu}(r, \varepsilon\tilde{X}_r)\,\mathrm{d}r + \int_{t_1}^s \mathrm{d}W_r^\top \overline{\sigma}(r, \varepsilon\tilde{X}_r, \overline{X}_r, Y_r, Z_r)$,
- $Y_s = \xi(\varepsilon\tilde{X}_T, \overline{X}_T) - \int_s^T f(r, \varepsilon\tilde{X}_r, \overline{X}_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^T \mathrm{d}W_r^\top Z_r$

where $s \in [t_1, T]$. We now differentiate w.r.t. $\tilde{x}$ and $\overline{x}$ using strong regularity and the chain rule of Lemma A.3.1:

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_s = I_N + \int_{t_1}^s \delta_r^{\tilde{\mu},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_r\,\mathrm{d}r + \int_{t_1}^s \mathrm{d}W_r^\top \delta_r^{\tilde{\sigma},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_r, \tag{5.20}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\overline{X}_s = \int_{t_1}^s \varepsilon\delta_r^{\overline{\mu},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_r\,\mathrm{d}r + \int_{t_1}^s \mathrm{d}W_r^\top \left(\varepsilon\delta_r^{\overline{\sigma},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_r + \delta_r^{\overline{\sigma},\overline{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\overline{X}_r + \delta_r^{\overline{\sigma},y} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}Y_r + \delta_r^{\overline{\sigma},z} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}Z_r\right), \tag{5.21}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}Y_s = \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}Y_T - \int_s^T \mathrm{d}W_r^\top \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}Z_r - \\ - \int_s^T \left(\varepsilon\delta_r^{f,\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_r + \delta_r^{f,\overline{x}+y}\left(\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\overline{X}_r + \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}Y_r\right) + \delta_r^{f,z} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}Z_r\right)\mathrm{d}r,\end{aligned} \tag{5.22}$$

$$\frac{\mathrm{d}}{\mathrm{d}\overline{x}}\overline{X}_s = 1 + \int_{t_1}^s \mathrm{d}W_r^\top \left(\delta_r^{\overline{\sigma},\overline{x}} \frac{\mathrm{d}}{\mathrm{d}\overline{x}}\overline{X}_r + \delta_r^{\overline{\sigma},y} \frac{\mathrm{d}}{\mathrm{d}\overline{x}}Y_r + \delta_r^{\overline{\sigma},z} \frac{\mathrm{d}}{\mathrm{d}\overline{x}}Z_r\right), \tag{5.23}$$

$$\frac{\mathrm{d}}{\mathrm{d}\overline{x}}Y_s = \frac{\mathrm{d}}{\mathrm{d}\overline{x}}Y_T - \int_s^T \mathrm{d}W_r^\top \frac{\mathrm{d}}{\mathrm{d}\overline{x}}Z_r - \int_s^T \left(\delta_r^{f,\overline{x}+y}\left(\frac{\mathrm{d}}{\mathrm{d}\overline{x}}\overline{X}_r + \frac{\mathrm{d}}{\mathrm{d}\overline{x}}Y_r\right) + \delta_r^{f,z} \frac{\mathrm{d}}{\mathrm{d}\overline{x}}Z_r\right)\mathrm{d}r, \tag{5.24}$$

a.s. for all $s \in [t_1, T]$, for almost all $(\tilde{x}, \overline{x}) \in \mathbb{R}^{N+1} \times \mathbb{R}$ with the following progressively measurable and bounded processes:

- $\delta^{\tilde{\mu},\tilde{x}}$, $\delta^{\tilde{\sigma},\tilde{x}}$, $\delta^{\overline{\mu},\tilde{x}}$ which are provided by Lemma A.3.1 and are bounded independently of t_1 and ε,
- $\delta^{\overline{\sigma},\tilde{x}} = \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\overline{\sigma}(\cdot, \tilde{X}, \overline{X}, Y, Z)$, $\delta^{f,\tilde{x}} = \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}f(\cdot, \tilde{X}, \overline{X}, Y, Z)$, which are also uniformly bounded,
- $\delta^{\overline{\sigma},\overline{x}} = \frac{\mathrm{d}}{\mathrm{d}\overline{x}}\overline{\sigma}(\cdot, \tilde{X}, \overline{X}, Y, Z)$, $\delta^{\overline{\sigma},y} = \frac{\mathrm{d}}{\mathrm{d}y}\overline{\sigma}(\cdot, \tilde{X}, \overline{X}, Y, Z)$ and $\delta^{\overline{\sigma},z} = \frac{\mathrm{d}}{\mathrm{d}z}\overline{\sigma}(\cdot, \tilde{X}, \overline{X}, Y, Z)$, which are uniformly bounded as well,
- $\delta^{f,\overline{x}+y} = \frac{\mathrm{d}}{\mathrm{d}(\overline{x}+y)}f(\cdot, \tilde{X}, \overline{X}, Y, Z)$ and $\delta^{f,z} = \frac{\mathrm{d}}{\mathrm{d}z}f(\cdot, \tilde{X}, \overline{X}, Y, Z)$, which are bounded but not necessarily uniformly in t_1, ε.

Uniform boundedness of $\delta^{\tilde{\mu},\tilde{x}}$, $\delta^{\tilde{\sigma},\tilde{x}}$, $\delta^{\overline{\mu},\tilde{x}}$, $\delta^{\overline{\sigma},\tilde{x}}$, $\delta^{f,\tilde{x}}$, $\delta^{\overline{\sigma},\overline{x}}$, $\delta^{\overline{\sigma},y}$, $\delta^{\overline{\sigma},z}$ is a consequence of the Lipschitz continuity assumptions we have made. Boundedness of $\delta^{f,\overline{x}+y}$ and $\delta^{f,z}$, however, follows from the structural assumptions on f together with the boundedness of Z.
Note also that:

- $\delta^{\tilde{\sigma},\tilde{x}}$ is $\mathbb{R}^{(d\times N)\times N}$ - valued and can also be interpreted as a vector $\left(\delta^{\tilde{\sigma},\tilde{x},i}\right)_{i=1,\dots,d}$, where $\delta^{\tilde{\sigma},\tilde{x},i}$ are $\mathbb{R}^{N\times N}$ - valued.

According to the structural assumptions for f we have:

- $\delta^{f,\overline{x}+y}$ is real-valued and non-negative,
- $\delta^{f,z}$ is an $\mathbb{R}^{1\times d}$-valued vector, where the first d_1 components vanish.

According to the structural assumptions for $\overline{\sigma}$ we have:

- $\delta^{\overline{\sigma},z}$ is an $\mathbb{R}^{d\times d}$ - valued bounded matrix consisting of an upper left $d_1 \times d_1$ block and a lower right $d_2 \times d_2$ block, such that all remaining components vanish and the operator norm of the matrix itself is bounded by $L_{\overline{\sigma},z} \leq 1$,

- $\delta^{\overline{\sigma},\overline{x}}$, $\delta^{\overline{\sigma},y}$ are $\mathbb{R}^d$ - valued vectors, for which the last d_2 components vanish.

Now, define

- $U_r := \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\overline{X}_r$, $V_r := \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}Y_r$, $\tilde{Z}_r := \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}Z_r$,

- $\hat{V}_r := \frac{\mathrm{d}}{\mathrm{d}\overline{x}}u(r, \tilde{X}_r, \overline{X}_r)$.

$\hat{V}$ is bounded by $K < L^{-1}_{\overline{\sigma},z}$: We assume without loss of generality that $\left|\frac{\mathrm{d}}{\mathrm{d}x}u\right| \leq K$ everywhere. As in the proof of Theorem 5.2.1, we can show that U is positive. Similarly we can assume that $\hat{V}$ is continuous in time. Define further:

- $\hat{U}_r := (U_r)^{-1} = \frac{1}{U_r}$,

- $\hat{Z}_r := \hat{U}_r\tilde{Z}_r - \hat{V}_r\left(\delta^{\overline{\sigma},\overline{x}}_r + \delta^{\overline{\sigma},y}_r\hat{V}_r + \delta^{\overline{\sigma},z}_r\tilde{Z}_r\hat{U}_r\right)$.

As in the proof of Theorem 5.2.1 we can deduce that $\hat{V}$ satisfies dynamics

$$\begin{aligned}\hat{V}_s = \hat{V}_T - \int_s^T \mathrm{d}W_r^\top \hat{Z}_r - \int_s^T \Bigg(&\delta^{f,\overline{x}+y}_r + \delta^{f,\overline{x}+y}_r\hat{V}_r + \delta^{f,z}_r\left(I_d - \hat{V}_r\delta^{\overline{\sigma},z}_r\right)^{-1}\hat{Z}_r - \\ &- \hat{Z}_r^\top\left(\delta^{\overline{\sigma},\overline{x}}_r + \hat{V}_r\delta^{\overline{\sigma},y}_r + \delta^{\overline{\sigma},z}_r\left(I_d - \hat{V}_r\delta^{\overline{\sigma},z}_r\right)^{-1}\left(\hat{Z}_r + \hat{V}_r\left(\delta^{\overline{\sigma},\overline{x}}_r + \delta^{\overline{\sigma},y}_r\hat{V}_r\right)\right)\right)\Bigg)\,\mathrm{d}r, \quad (5.25)\end{aligned}$$

which follows from (5.11) taking into account that we switched from $\mathbb{R}^{1\times d}$ - to $\mathbb{R}^d$ - valued vectors. Due to the hypotheses of the theorem $\hat{V}_r$ and $\left(I_d - \hat{V}_r\delta^{\overline{\sigma},z}_r\right)^{-1}$ are uniformly bounded by constants, which do not depend on t_1!
Furthermore, according to the Itô formula and the dynamics of $\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}$ given by (5.20) its matrix inverse $\tilde{U}_r := \left(\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_r\right)^{-1}$ exists and has dynamics

$$\tilde{U}_s = I_N + \int_{t_1}^s \tilde{U}_r\left(\sum_{i=1}^d \delta^{\tilde{\sigma},\tilde{x},i}_r\delta^{\tilde{\sigma},\tilde{x},i}_r - \delta^{\tilde{\mu},\tilde{x}}_r\right)\mathrm{d}r - \sum_{i=1}^d\int_{t_1}^s \mathrm{d}W^i_r\tilde{U}_r\delta^{\tilde{\sigma},\tilde{x},i}_r, \quad (5.26)$$

a.s. for all $s \in [t_1, T]$. In particular, we can assume that $\tilde{U}$ is continuous in time.
Using the chain rule of Lemma A.3.1 we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}Y_s = \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\left(u(s,\tilde{X}_s,\overline{X}_s)\right) &= \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}u(s,\tilde{X}_s,\overline{X}_s)\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_s + \frac{\mathrm{d}}{\mathrm{d}\overline{x}}u(s,\tilde{X}_s,\overline{X}_s)\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\overline{X}_s = \\ &= \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}u(s,\tilde{X}_s,\overline{X}_s)\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_s + \hat{V}_s\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\overline{X}_s.\end{aligned}$$

Now, define

- $R_s := \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}Y_s - \hat{V}_s\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\overline{X}_s = \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}u(s,\tilde{X}_s,\overline{X}_s)\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_s$ and

- $\tilde{R}_s := R_s\tilde{U}_s = \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}u(s,\tilde{X}_s,\overline{X}_s)$.

Using the Itô formula we can deduce the dynamics of R and then of $\tilde{R}$. Let us first deal with R. We will use (5.22), (5.21) and (5.25):

$$R_s = R_T - \int_s^T \mathrm{d}W_r^\top \check{Z}_r - \int_s^T H_r \, \mathrm{d}r,$$

where

$$\check{Z}_r := \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Z_r - \hat{Z}_r \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r - \hat{V}_r \left(\varepsilon \delta_r^{\overline{\sigma},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r + \delta_r^{\overline{\sigma},\overline{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r + \delta_r^{\overline{\sigma},y} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Y_r + \delta_r^{\overline{\sigma},z} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Z_r \right)$$

and

$$\begin{aligned} H_r := {} & \varepsilon \delta_r^{f,\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r + \delta_r^{f,\overline{x}+y} \left(\frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r + \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Y_r \right) + \delta_r^{f,z} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Z_r - \\ & - \Bigg\{ \delta_r^{f,\overline{x}+y} + \delta_r^{f,\overline{x}+y} \hat{V}_r + \delta_r^{f,z} \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z} \right)^{-1} \hat{Z}_r - \\ & - \hat{Z}_r^\top \left(\delta_r^{\overline{\sigma},\overline{x}} + \hat{V}_r \delta_r^{\overline{\sigma},y} + \delta_r^{\overline{\sigma},z} \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z} \right)^{-1} \left(\hat{Z}_r + \hat{V}_r \left(\delta_r^{\overline{\sigma},\overline{x}} + \delta_r^{\overline{\sigma},y} \hat{V}_r \right) \right) \right) \Bigg\} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r - \\ & - \hat{V}_r \varepsilon \delta_r^{\overline{\mu},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r - \hat{Z}_r^\top \left(\varepsilon \delta_r^{\overline{\sigma},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r + \delta_r^{\overline{\sigma},\overline{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r + \delta_r^{\overline{\sigma},y} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Y_r + \delta_r^{\overline{\sigma},z} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Z_r \right). \end{aligned}$$

In the above expression the marked terms effectively cancel out and we obtain:

$$\begin{aligned} H_r = {} & \varepsilon \delta_r^{f,\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r + \delta_r^{f,\overline{x}+y} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Y_r + \delta_r^{f,z} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Z_r - \Bigg\{ \delta_r^{f,\overline{x}+y} \hat{V}_r + \delta_r^{f,z} \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z} \right)^{-1} \hat{Z}_r - \\ & - \hat{Z}_r^\top \left(\hat{V}_r \delta_r^{\overline{\sigma},y} + \delta_r^{\overline{\sigma},z} \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z} \right)^{-1} \left(\hat{Z}_r + \hat{V}_r \left(\delta_r^{\overline{\sigma},\overline{x}} + \delta_r^{\overline{\sigma},y} \hat{V}_r \right) \right) \right) \Bigg\} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r - \\ & - \hat{V}_r \varepsilon \delta_r^{\overline{\mu},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r - \hat{Z}_r^\top \left(\varepsilon \delta_r^{\overline{\sigma},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r + \delta_r^{\overline{\sigma},y} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Y_r + \delta_r^{\overline{\sigma},z} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Z_r \right). \end{aligned}$$

In the above expression the marked terms can be effectively merged using $\frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Y_r - \hat{V}_s \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r = R_r$, so we can further simplify:

$$\begin{aligned} H_r = {} & \varepsilon \delta_r^{f,\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r + \delta_r^{f,\overline{x}+y} R_r + \delta_r^{f,z} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Z_r - \delta_r^{f,z} \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z} \right)^{-1} \hat{Z}_r \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r + \\ & + \hat{Z}_r^\top \left(\delta_r^{\overline{\sigma},z} \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z} \right)^{-1} \left(\hat{Z}_r + \hat{V}_r \left(\delta_r^{\overline{\sigma},\overline{x}} + \delta_r^{\overline{\sigma},y} \hat{V}_r \right) \right) \right) \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r - \\ & - \hat{V}_r \varepsilon \delta_r^{\overline{\mu},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r - \hat{Z}_r^\top \left(\varepsilon \delta_r^{\overline{\sigma},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r + \delta_r^{\overline{\sigma},y} R_r + \delta_r^{\overline{\sigma},z} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Z_r \right). \end{aligned} \tag{5.27}$$

Now, using the definition of $\check{Z}$ we can write

$$\check{Z}_r + \hat{Z}_r \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r + \hat{V}_r \left(\varepsilon \delta_r^{\overline{\sigma},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r + \delta_r^{\overline{\sigma},\overline{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r + \delta_r^{\overline{\sigma},y} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Y_r \right) = \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Z_r - \hat{V}_r \delta_r^{\overline{\sigma},z} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Z_r,$$

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Z_r = \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z} \right)^{-1} \left(\check{Z}_r + \hat{Z}_r \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r + \hat{V}_r \left(\varepsilon \delta_r^{\overline{\sigma},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r + \delta_r^{\overline{\sigma},\overline{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r + \delta_r^{\overline{\sigma},y} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Y_r \right) \right) =$$

$$= \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z}\right)^{-1} \left(\check{Z}_r + \hat{V}_r \varepsilon \delta_r^{\overline{\sigma},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r + \hat{Z}_r \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r + \hat{V}_r \left(\delta_r^{\overline{\sigma},\overline{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r + \delta_r^{\overline{\sigma},y} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Y_r \right)\right).$$

We again use $\frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Y_r = R_r + \hat{V}_s \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Z_r &= \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z}\right)^{-1} \cdot \\
&\qquad \cdot \left(\check{Z}_r + \hat{V}_r \varepsilon \delta_r^{\overline{\sigma},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r + \hat{Z}_r \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r + \hat{V}_r \left(\delta_r^{\overline{\sigma},\overline{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r + \delta_r^{\overline{\sigma},y} R_r + \delta_r^{\overline{\sigma},y} \hat{V}_s \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r \right)\right) = \\
&= \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z}\right)^{-1} \cdot \\
&\qquad \cdot \left(\check{Z}_r + \hat{V}_r \varepsilon \delta_r^{\overline{\sigma},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r + \hat{V}_r \delta_r^{\overline{\sigma},y} R_r + \hat{Z}_r \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r + \hat{V}_r \left(\delta_r^{\overline{\sigma},\overline{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r + \delta_r^{\overline{\sigma},y} \hat{V}_s \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r \right)\right) = \\
&= \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z}\right)^{-1} \left(\check{Z}_r + \hat{V}_r \varepsilon \delta_r^{\overline{\sigma},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r + \hat{V}_r \delta_r^{\overline{\sigma},y} R_r + \left(\hat{Z}_r + \hat{V}_r \left(\delta_r^{\overline{\sigma},\overline{x}} + \delta_r^{\overline{\sigma},y} \hat{V}_s \right)\right) \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r \right),
\end{aligned} \tag{5.28}$$

where in the last step we used the distributive law.
Now, let us plug (5.28) into (5.27) at the first place where $\frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Z_r$ appears: Remember that the first d_1 components of $\delta^{f,z}$ and the last d_2 components of $\delta^{\overline{\sigma},\overline{x}}$ and $\delta^{\overline{\sigma},y}$ vanish. So, the expressions

$$\delta_r^{f,z} \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z}\right)^{-1} \cdot \delta_r^{\overline{\sigma},\overline{x}} \qquad \text{and} \qquad \delta_r^{f,z} \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z}\right)^{-1} \cdot \delta_r^{\overline{\sigma},y}$$

vanish and we end up with:

$$\begin{aligned}
H_r &= \varepsilon \delta_r^{f,\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r + \delta_r^{f,\overline{x}+y} R_r + \delta_r^{f,z} \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z}\right)^{-1} \left(\check{Z}_r + \hat{V}_r \varepsilon \delta_r^{\overline{\sigma},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r + \hat{Z}_r \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r \right) - \\
&- \delta_r^{f,z} \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z}\right)^{-1} \hat{Z}_r \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r + \hat{Z}_r^\top \left(\delta_r^{\overline{\sigma},z} \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z}\right)^{-1} \left(\hat{Z}_r + \hat{V}_r \left(\delta_r^{\overline{\sigma},\overline{x}} + \delta_r^{\overline{\sigma},y} \hat{V}_r \right)\right)\right) \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r - \\
&\qquad - \hat{V}_r \varepsilon \delta_r^{\overline{\mu},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r - \hat{Z}_r^\top \left(\varepsilon \delta_r^{\overline{\sigma},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r + \delta_r^{\overline{\sigma},y} R_r + \delta_r^{\overline{\sigma},z} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Z_r \right).
\end{aligned}$$

Note that the marked terms above effectively cancel out, so we obtain:

$$\begin{aligned}
H_r &= \varepsilon \delta_r^{f,\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r + \delta_r^{f,\overline{x}+y} R_r + \delta_r^{f,z} \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z}\right)^{-1} \left(\check{Z}_r + \hat{V}_r \varepsilon \delta_r^{\overline{\sigma},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r \right) + \\
&\qquad + \hat{Z}_r^\top \delta_r^{\overline{\sigma},z} \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z}\right)^{-1} \left(\hat{Z}_r + \hat{V}_r \left(\delta_r^{\overline{\sigma},\overline{x}} + \delta_r^{\overline{\sigma},y} \hat{V}_r \right)\right) \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r - \\
&\qquad - \hat{V}_r \varepsilon \delta_r^{\overline{\mu},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r - \hat{Z}_r^\top \left(\varepsilon \delta_r^{\overline{\sigma},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r + \delta_r^{\overline{\sigma},y} R_r + \delta_r^{\overline{\sigma},z} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Z_r \right).
\end{aligned} \tag{5.29}$$

But according to (5.28) we can write

$$\begin{aligned}
\hat{Z}_r^\top \delta_r^{\overline{\sigma},z} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} Z_r &= \hat{Z}_r^\top \delta_r^{\overline{\sigma},z} \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z}\right)^{-1} \left(\check{Z}_r + \hat{V}_r \varepsilon \delta_r^{\overline{\sigma},\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r + \hat{V}_r \delta_r^{\overline{\sigma},y} R_r \right) + \\
&\qquad + \hat{Z}_r^\top \delta_r^{\overline{\sigma},z} \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z}\right)^{-1} \left(\hat{Z}_r + \hat{V}_r \left(\delta_r^{\overline{\sigma},\overline{x}} + \delta_r^{\overline{\sigma},y} \hat{V}_s \right)\right) \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \overline{X}_r,
\end{aligned}$$

so plugging this into (5.29) leads to:

$$
\begin{aligned}
H_r = \varepsilon\delta_r^{f,\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_r + \delta_r^{f,\overline{x}+y} R_r + \delta_r^{f,z}\left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1}\left(\check{Z}_r + \hat{V}_r\varepsilon\delta_r^{\overline{\sigma},\tilde{x}}\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_r\right) - \\
- \hat{V}_r\varepsilon\delta_r^{\overline{\mu},\tilde{x}}\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_r - \hat{Z}_r^\top\left(\varepsilon\delta_r^{\overline{\sigma},\tilde{x}}\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_r + \delta_r^{\overline{\sigma},y}R_r\right) - \\
- \hat{Z}_r^\top\delta_r^{\overline{\sigma},z}\left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1}\left(\check{Z}_r + \hat{V}_r\,\varepsilon\delta_r^{\overline{\sigma},\tilde{x}}\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_r + \hat{V}_r\,\delta_r^{\overline{\sigma},y}R_r\right).
\end{aligned}
$$

Note that we have $I_d + \hat{V}_r\delta_r^{\overline{\sigma},z}\left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1} = \left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1}$, which can be easily verified by multiplying both sides of this equation with $I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}$ from the right. Using this relationship we can combine the marked terms above, obtaining:

$$
\begin{aligned}
H_r = \varepsilon\delta_r^{f,\tilde{x}} \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_r + \delta_r^{f,\overline{x}+y} R_r + \delta_r^{f,z}\left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1}\left(\check{Z}_r + \hat{V}_r\varepsilon\delta_r^{\overline{\sigma},\tilde{x}}\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_r\right) - \\
- \hat{V}_r\varepsilon\delta_r^{\overline{\mu},\tilde{x}}\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_r - \hat{Z}_r^\top\left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1}\left(\varepsilon\delta_r^{\overline{\sigma},\tilde{x}}\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_r + \delta_r^{\overline{\sigma},y}R_r\right) - \\
- \hat{Z}_r^\top\delta_r^{\overline{\sigma},z}\left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1}\check{Z}_r.
\end{aligned}
$$

Using the distributive law we obtain

$$
\begin{aligned}
H_r = \left(\varepsilon\delta_r^{f,\tilde{x}} + \delta_r^{f,z}\left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1}\hat{V}_r\varepsilon\,\delta_r^{\overline{\sigma},\tilde{x}} - \hat{V}_r\varepsilon\delta_r^{\overline{\mu},\tilde{x}} - \hat{Z}_r^\top\left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1}\varepsilon\,\delta_r^{\overline{\sigma},\tilde{x}}\right)\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_r + \\
+ \left(\delta_r^{f,\overline{x}+y} - \hat{Z}_r^\top\left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1}\delta_r^{\overline{\sigma},y}\right)R_r + \\
+ \delta_r^{f,z}\left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1}\check{Z}_r - \hat{Z}_r^\top\delta_r^{\overline{\sigma},z}\left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1}\check{Z}_r,
\end{aligned}
$$

which further simplifies to

$$
\begin{aligned}
H_r = \varepsilon\left(\delta_r^{f,\tilde{x}} + \left(\hat{V}_r\delta_r^{f,z} - \hat{Z}_r^\top\right)\left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1}\delta_r^{\overline{\sigma},\tilde{x}} - \hat{V}_r\delta_r^{\overline{\mu},\tilde{x}}\right)\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\tilde{X}_r + \\
+ \left(\delta_r^{f,\overline{x}+y} - \hat{Z}_r^\top\left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1}\delta_r^{\overline{\sigma},y}\right)R_r + \\
+ \left(\delta_r^{f,z} - \hat{Z}_r^\top\delta_r^{\overline{\sigma},z}\right)\left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1}\check{Z}_r.
\end{aligned}
$$

Let us now deduce the dynamics of $\tilde{R} = R\tilde{U}$. We use the dynamics of R we just obtained, as well as (5.26), which describes dynamics of $\tilde{U}$:

$$
\tilde{R}_s = \tilde{R}_T - \int_s^T \mathrm{d}W_r^\top \dot{Z}_r - \int_s^T G_r\,\mathrm{d}r,
$$

where $\dot{Z}$ is $\mathbb{R}^{d\times N}$ - valued and can be written as $\left(\dot{Z}^i\right)_{i=1,\ldots,d}$ such that

$$
\dot{Z}_r^i = \check{Z}_r^i\tilde{U}_r - R_r\tilde{U}_r\delta_r^{\tilde{\sigma},\tilde{x},i} = \check{Z}_r^i\tilde{U}_r - \tilde{R}_r\delta_r^{\tilde{\sigma},\tilde{x},i},
$$

for $i = 1, \ldots, d$, and

$$G_r = H_r \tilde{U}_r + R_r \tilde{U}_r \left(\sum_{i=1}^{d} \delta_r^{\tilde{\sigma},\tilde{x},i} \delta_r^{\tilde{\sigma},\tilde{x},i} - \delta_r^{\tilde{\mu},\tilde{x}} \right) - \sum_{i=1}^{d} \check{Z}_r^i \tilde{U}_r \, \delta_r^{\tilde{\sigma},\tilde{x},i} =$$

$$= H_r \tilde{U}_r + \tilde{R}_r \left(\sum_{i=1}^{d} \delta_r^{\tilde{\sigma},\tilde{x},i} \delta_r^{\tilde{\sigma},\tilde{x},i} - \delta_r^{\tilde{\mu},\tilde{x}} \right) - \sum_{i=1}^{d} \left(\dot{Z}_r^i + \tilde{R}_r \delta_r^{\tilde{\sigma},\tilde{x},i} \right) \delta_r^{\tilde{\sigma},\tilde{x},i} = H_r \tilde{U}_r - \tilde{R}_r \delta_r^{\tilde{\mu},\tilde{x}} - \sum_{i=1}^{d} \dot{Z}_r^i \delta_r^{\tilde{\sigma},\tilde{x},i}.$$

By plugging in the structure of H and using $\left(\frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \tilde{X}_r \right) \tilde{U}_r = I_N$ we obtain

$$\begin{aligned} G_r = \varepsilon \Bigg(\delta_r^{f,\tilde{x}} + \left(\hat{V}_r \delta_r^{f,z} - \hat{Z}_r^\top \right) \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z} \right)^{-1} \delta_r^{\overline{\sigma},\tilde{x}} - \hat{V}_r \delta_r^{\overline{\mu},\tilde{x}} \Bigg) + \\ + \left(\delta_r^{f,\overline{x}+y} - \hat{Z}_r^\top \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z} \right)^{-1} \delta_r^{\overline{\sigma},y} \right) \tilde{R}_r + \\ + \left(\delta_r^{f,z} - \hat{Z}_r^\top \delta_r^{\overline{\sigma},z} \right) \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z} \right)^{-1} \check{Z}_r \tilde{U}_r - \tilde{R}_r \, \delta_r^{\tilde{\mu},\tilde{x}} - \sum_{i=1}^{d} \dot{Z}_r^i \delta_r^{\tilde{\sigma},\tilde{x},i}. \end{aligned}$$

Knowing $\check{Z}_r^i \tilde{U}_r = \dot{Z}_r^i + \tilde{R}_r \delta_r^{\tilde{\sigma},\tilde{x},i}$ we obtain

$$\begin{aligned} G_r = \varepsilon \Bigg(\delta_r^{f,\tilde{x}} + \left(\hat{V}_r \delta_r^{f,z} - \hat{Z}_r^\top \right) \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z} \right)^{-1} \delta_r^{\overline{\sigma},\tilde{x}} - \hat{V}_r \delta_r^{\overline{\mu},\tilde{x}} \Bigg) + \\ + \tilde{R}_r \left(-\delta_r^{\tilde{\mu},\tilde{x}} + \left(\delta_r^{f,\overline{x}+y} - \hat{Z}_r^\top \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z} \right)^{-1} \delta_r^{\overline{\sigma},y} \right) I_N \right) + \\ + \left(\delta_r^{f,z} - \hat{Z}_r^\top \delta_r^{\overline{\sigma},z} \right) \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z} \right)^{-1} \dot{Z}_r + \\ + \sum_{i=1}^{d} \left(\left(\delta_r^{f,z} - \hat{Z}_r^\top \delta_r^{\overline{\sigma},z} \right) \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z} \right)^{-1} \right)^i \tilde{R}_r \, \delta_r^{\tilde{\sigma},\tilde{x},i} - \sum_{i=1}^{d} \dot{Z}_r^i \delta_r^{\tilde{\sigma},\tilde{x},i}, \end{aligned}$$

where z^i refers to the i - th component of a vector $z \in \mathbb{R}^{1 \times d}$.
We can rewrite using distributive law:

$$\begin{aligned} G_r = \varepsilon \Bigg(\delta_r^{f,\tilde{x}} + \left(\hat{V}_r \delta_r^{f,z} - \hat{Z}_r^\top \right) \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z} \right)^{-1} \delta_r^{\overline{\sigma},\tilde{x}} - \hat{V}_r \delta_r^{\overline{\mu},\tilde{x}} \Bigg) + \\ + \tilde{R}_r \Bigg\{ - \delta_r^{\tilde{\mu},\tilde{x}} + \left(\delta_r^{f,\overline{x}+y} - \hat{Z}_r^\top \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z} \right)^{-1} \delta_r^{\overline{\sigma},y} \right) I_N + \\ + \sum_{i=1}^{d} \left(\left(\delta_r^{f,z} - \hat{Z}_r^\top \delta_r^{\overline{\sigma},z} \right) \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z} \right)^{-1} \right)^i \delta_r^{\tilde{\sigma},\tilde{x},i} \Bigg\} + \\ + \left(\delta_r^{f,z} - \hat{Z}_r^\top \delta_r^{\overline{\sigma},z} \right) \left(I_d - \hat{V}_r \delta_r^{\overline{\sigma},z} \right)^{-1} \dot{Z}_r - \sum_{i=1}^{d} \dot{Z}_r^i \delta_r^{\tilde{\sigma},\tilde{x},i}. \end{aligned} \tag{5.30}$$

Now, remember that Y satisfies

$$Y_s = Y_T - \int_s^T \mathrm{d}W_r^\top Z_r - \int_s^T f(r, \tilde{X}_r, \overline{X}_r, Y_r, Z_r) \, \mathrm{d}r, \qquad r \in [t_1, T].$$

Also, $q_r := f(r, \tilde{X}_r, \overline{X}_r, Y_r, 0)$ is bounded by $\|f(\cdot,\cdot,\cdot,\cdot,0)\|_\infty$, and furthermore the difference

$$f(r, \tilde{X}_r, \overline{X}_r, Y_r, Z_r) - f(r, \tilde{X}_r, \overline{X}_r, Y_r, 0) = p_r Z_r,$$

where the bounded process p is defined via

$$p_r := \left(\frac{1}{|Z_r|^2} \left(f(r, \tilde{X}_r, \overline{X}_r, Y_r, Z_r) - f(r, \tilde{X}_r, \overline{X}_r, Y_r, 0) \right) Z_r^\top \right),$$

is bounded by $C(1 + |Z_r|)$ due to our requirements for $\frac{\mathrm{d}}{\mathrm{d}z} f$.
The backward equation $Y_s = Y_T - \int_s^T \mathrm{d}W_r^\top Z_r - \int_s^T q_r + p_r Z_r \, \mathrm{d}r$ together with the boundedness of $Y_T = \xi(\varepsilon \tilde{X}_T, \overline{X}_T)$ imply:

- Y is uniformly bounded by $\|\xi\|_\infty + T\|f(\cdot,\cdot,\cdot,\cdot,0)\|_\infty$ (see Lemma A.1.10, which is applicable since Z is bounded),
- Z and therefore $\delta^{f,z} = \frac{\mathrm{d}}{\mathrm{d}z} f(\cdot, \tilde{X}, \overline{X}, Y, Z)$ are BMO - processes with a $BMO(\mathbb{P})$ - norm controlled independently of t_1 and ε (see Theorem A.1.11). ✓

Also, due to

- the dynamics of $\hat{V}$ given by (5.25),
- the uniform boundedness of $\hat{V}$ and $\left(I_d - \hat{V} \delta^{\overline{\sigma},z} \right)^{-1}$,
- our requirements for $\frac{\mathrm{d}}{\mathrm{d}z} f$ and $\frac{\mathrm{d}}{\mathrm{d}(\overline{x}+y)} f$

Theorem A.1.11 is applicable to (5.25) and we have that $\hat{Z}$ is also a BMO - process with a $BMO(\mathbb{P})$ - norm controlled independently of t_1 and ε. ✓

Using (5.30) the process $\tilde{R}$ has dynamics

$$\tilde{R}_s = \tilde{R}_T - \int_s^T \mathrm{d}W_r^\top \dot{Z}_r - \int_s^T \left(\varepsilon \alpha_r + \tilde{R}_r \left(\delta_r^{f,\overline{x}+y} I_N + \beta_r \right) + \mu \dot{Z}_r + \sum_{i=1}^d \dot{Z}_r^i \gamma_r^i \right) \mathrm{d}r,$$

where

- α is an $\mathbb{R}^{1\times N}$-valued BMO process,
- β is an $\mathbb{R}^{N\times N}$-valued BMO process,
- μ is an $\mathbb{R}^{1\times d}$-valued BMO process and
- γ^i, $i = 1, \ldots, d$ are bounded progressive $\mathbb{R}^{N\times N}$-valued processes,

such that the $BMO(\mathbb{P})$ - norms of α, β, μ and supremum norms of γ^i can be controlled independently of t_1 and ε.
Also, note the relationship $\tilde{R}_T = \varepsilon \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \xi(\varepsilon \tilde{X}_T, \overline{X}_T)$, which is a direct consequence of the terminal condition $u(T, \tilde{x}, \overline{x}) = \xi(\varepsilon \tilde{x}, \overline{x})$. So, $\tilde{R}_T$ is bounded by $\varepsilon L_{\xi,\tilde{x}}$.
We know that $\tilde{R}_s = \frac{\mathrm{d}}{\mathrm{d}\tilde{x}} u(s, \tilde{X}_s, \overline{X}_s)$ is a bounded process but not necessarily bounded independently of t_1, ε (at this point), however, we can now apply Lemma A.1.7 to obtain

$$\|\tilde{R}\|_\infty \le C\varepsilon L_{\xi,\tilde{x}} + C\varepsilon \|\alpha\|_{BMO(\mathbb{P})},$$

where $C \in (0,\infty)$ depends only on T, $\|\mu\|_{BMO(\mathbb{P})}$, $\|\beta\|_{BMO(\mathbb{P})}$ and $\|\gamma\|_\infty$ and is monotonically increasing in these values.

This shows that for $\varepsilon > 0$ small enough $L_{u(t_1,\cdot),(\tilde{x},\overline{x})^\top} \le K + \varepsilon \tilde{C} < L_{\overline{\sigma},z}^{-1}$ will hold independently of t_1, where $\tilde{C}$ is a constant, which does not depend on t_1 and ε. This contradicts the statement of Lemma 5.3.11. Therefore, the assumption $I_{\max}^M = (t_{\min}^M, T]$ was wrong and so, $I_{\max}^M = [0,T]$ for $\varepsilon > 0$ small enough is proven. □

5.3.3 Main result

Now, let us apply the above abstract result to solve the actual FBSDE (5.4) under certain conditions:

We want to investigate the solvability of the forward backward system given by the forward equation

$$\tilde{X}_t = \tilde{x} + \int_0^t \tilde{\mu}(s, \tilde{X}_s)\,\mathrm{d}s + \int_0^t \mathrm{d}W_s^\top \tilde{\sigma}(s, \tilde{X}_s),$$

$$X_t = x - \int_0^t \left(\mathrm{d}W_s + \pi_1(\theta(s, \tilde{X}_s))\,\mathrm{d}s\right)^\top \left(\pi_1(\theta(s, \tilde{X}_s))\frac{U'}{U''}(X_s + Y_s) + \pi_1(Z_s)\right)$$

and the backward equation

$$\begin{aligned}Y_t = H(\tilde{X}_T, X_T) &- \int_t^T \left(\mathrm{d}W_s + \pi_1(\theta(s, \tilde{X}_s))\,\mathrm{d}s\right)^\top Z_s - \\ &- \int_t^T \left(|\pi_1(\theta(s, \tilde{X}_s))|^2 \frac{U'}{U''}\left(1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2}\right)(X_s + Y_s) - \frac{1}{2}|\pi_2(Z_s)|^2 \cdot \frac{U^{(3)}}{U''}(X_s + Y_s)\right)\mathrm{d}s,\end{aligned}$$

$t \in [0, T]$, where $\tilde{x} \in \mathbb{R}^N$, $N \in \mathbb{R}$, $x \in \mathbb{R}$,

$$U(x) := -\int_x^\infty \int_y^\infty \exp(-\kappa(z))\,\mathrm{d}z\,\mathrm{d}y$$

with some $\kappa : \mathbb{R} \to \mathbb{R}$ satisfying:

- κ is twice differentiable,
- $0 < \inf_{x\in\mathbb{R}} \kappa'(x) \le \sup_{x\in\mathbb{R}} \kappa'(x) < \infty$ and
- $0 \le \inf_{x\in\mathbb{R}} \kappa''(x) \le \sup_{x\in\mathbb{R}} \kappa''(x) < \infty$.

According to Remark 5.2.4 the function κ' can be interpreted as the local risk aversion for the utility function U.
So, the problem is about finding progressively measurable processes $\tilde{X}, X, Y, Z$ s.t.

- $\tilde{X}$ is $\mathbb{R}^N$ - valued,
- X and Y are $\mathbb{R}$ - valued,
- Z is $\mathbb{R}^d$ - valued

and the above FBSDE is satisfied.
We assume that

- $\tilde{\mu}, \tilde{\sigma} : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N, \mathbb{R}^{d\times N}$ are measurable and Lipschitz continuous in the second component with Lipschitz constants $L_{\tilde{\mu},\tilde{x}}$, $L_{\tilde{\sigma},\tilde{x}}$ and such that $\|\tilde{\mu}(\cdot, 0)\|_\infty, \|\tilde{\sigma}\|_\infty < \infty$,
- $\theta : [0, T] \times \mathbb{R}^N \to \mathbb{R}^d$ is measurable, bounded and differentiable in the second component everywhere with a uniformly bounded derivative,
- $H : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is bounded and Lipschitz continuous in both components with Lipschitz constants $L_{H,\tilde{x}}$, $L_{H,x}$,
- $L_{H,x} < 1$, where $x \in \mathbb{R}$ refers to the second component.

Theorem 5.3.13. *Under these conditions the above problem has a unique solution $\tilde{X}, X, Y, Z$ on $[0,T]$ satisfying $\|Z\|_\infty < \infty$.*
Furthermore, the problem can be reduced to an MLLC problem with $I^M_{\max} = [0,T]$.

Proof. Note that the forward equation for $\tilde{X}$ has a unique solution which can be obtained independently of the other parts of the problem, since $\tilde{X}$ satisfies a rather standard SDE. So, our task is really about establishing existence and uniqueness of X, Y, Z.
We define a Brownian motion with drift B via

$$B_s = W_s + \int_0^s \pi_1(\theta(r, \tilde{X}_r))\,\mathrm{d}r, \quad s \in [0,T].$$

Note that B is a Brownian motion under some probability measure $\mathbb{Q} \sim \mathbb{P}$. $\tilde{X}$ has dynamics

$$\tilde{X}_t = \tilde{x} + \int_0^t \left(\tilde{\mu} - \pi_1(\theta)^\top \tilde{\sigma}\right)(r, \tilde{X}_r)\,\mathrm{d}r + \int_0^t \mathrm{d}B_r^\top \tilde{\sigma}(r, \tilde{X}_r),$$

which describes a uniquely solvable Lipschitz problem, so $\tilde{X}$ is adapted w.r.t. the filtration generated by B (and augmented by $\mathcal{F}_0$), which in turn implies that $W = B - \int_0^\cdot \pi_1(\theta(r, \tilde{X}_r)\,\mathrm{d}r$ is adapted w.r.t. the filtration generated by B as well. So, W and B generate the same filtration $(\mathcal{F}_t)_{t\in[0,T]}$.
We now introduce a slightly modified problem: For $\varepsilon > 0$ consider the system given by the forward equation

$$\begin{aligned}\check{X}_s &= \check{x} + \int_t^s \frac{1}{\varepsilon}\left(\tilde{\mu} - \pi_1(\theta)^\top \tilde{\sigma}\right)(r, \varepsilon\check{X}_r)\,\mathrm{d}r + \int_t^s \mathrm{d}B_r^\top \frac{1}{\varepsilon}\tilde{\sigma}(r, \varepsilon\check{X}_r), \\ X_s &= x - \int_t^s \mathrm{d}B_r^\top \left(\pi_1(\theta(r, \varepsilon\check{X}_r))\frac{U'}{U''}(X_r + Y_r) + \pi_1(Z_r)\right),\end{aligned} \tag{5.31}$$

and the backward equation

$$\begin{aligned}Y_s = {} & H(\varepsilon\check{X}_T, X_T) - \int_s^T \mathrm{d}B_r^\top Z_r - \\ & - \int_s^T \left(|\pi_1(\theta(r, \varepsilon\check{X}_r))|^2 \frac{U'}{U''}\left(1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2}\right)(X_r + Y_r) - \frac{1}{2}|\pi_2(Z_r)|^2 \cdot \frac{U^{(3)}}{U''}(X_r + Y_r)\right)\mathrm{d}r,\end{aligned} \tag{5.32}$$

$s \in [t,T]$. This new forward-backward system is completely equivalent to the preceding one in the sense that $\tilde{X}, X, Y, Z$ solve the initial system if and only if $\check{X} := \frac{1}{\varepsilon}\tilde{X}, X, Y, Z$ solve the new system. So, it remains to show, that for some $\varepsilon > 0$ the new system will have a unique solution with bounded Z. For that purpose we apply Theorem 5.3.12 to show that for the above problem $I^M_{\max} = [0,T]$ will hold for some $\varepsilon > 0$. Let us first check, that the system satisfies the structural requirements of Theorem 5.3.12:
Firstly, observe the following properties of U:

- According to Remark 5.2.4 $\frac{U'}{U''}$, $\frac{U''}{U'}$ and $\frac{U^{(3)}}{U''}$ are bounded.
- $(\ln(-U''))'' = -\kappa''$ is non-positive and bounded.
- U'' is point-wise negative.
- $\frac{U'}{U''}$, $\frac{U'}{U''}\left(1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2}\right)$ and $\frac{U^{(3)}}{U''}$ are bounded and Lipschitz continuous according to the proof of Theorem 5.2.2.
- $\left(\frac{U'}{U''}\left(1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2}\right)\right)' \geq 0$ and $\left(\frac{U^{(3)}}{U''}\right)' \leq 0$ according to the proof of Theorem 5.2.2.

Using the notation of the previous section the parameter functions $\overline{\mu}, \overline{\sigma}, f$ implied by the above problem (5.31), (5.32) satisfy:

- $\overline{\mu}$ vanishes,
- $\overline{\sigma}$ and f are differentiable in $\check{x}, x, y, z$ such that all of the partial derivatives are uniformly bounded except for $\frac{\mathrm{d}}{\mathrm{d}(x+y)} f$ and $\frac{\mathrm{d}}{\mathrm{d}z} f$. This boundedness comes from boundedness and Lipschitz continuity of θ together with the aforementioned properties of U.

The generator f of the backward equation satisfies the structural requirements of Theorem 5.3.12:

- It is a function of $s, \tilde{x}, x+y, \pi_2(z)$ and has quadratic growth in $\pi_2(z)$, while
- its derivative w.r.t. z has linear growth in $\pi_2(z)$ and
- its derivative w.r.t. $x+y$ has quadratic growth in $\pi_2(z)$. It is also non-negative according to the aforementioned properties of U. Finally
- $f(\cdot,\cdot,\cdot,\cdot,0)$ is uniformly bounded.

The parameter functions $\overline{\sigma}$ and H also have the structure required by Theorem 5.3.12: Note here that $\pi_2(\overline{\sigma}) = 0$ and $\overline{\sigma}$ is a function of $s, \check{x}, x+y, \pi_1(z)$. Also, $L_{\overline{\sigma},z} = 1$ such that $L_{H,x} < 1 = L_{\sigma,z}^{-1}$.

In order to apply Theorem 5.3.12 we merely need to control $\frac{\mathrm{d}}{\mathrm{d}x} u$ uniformly for every weakly regular Markovian decoupling field $u : [t,T] \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ to the above problem for small $\varepsilon > 0$. This control has to be independent of u, t and ε.
For this purpose we seek to control $\hat{V}_r := \frac{\mathrm{d}}{\mathrm{d}x} u(r, \check{X}_r, X_r)$, $r \in [t,T]$. Note that w.l.o.g. $\hat{V}$ is bounded by $L_{u,(\tilde{x},x)} < L_{\overline{\sigma},z}^{-1} = 1$. Using notations from the proof of Theorem 5.3.12 we have similarly to (5.25):

$$\hat{V}_s = \hat{V}_T - \int_s^T \mathrm{d}B_r^\top \hat{Z}_r - \int_s^T \Bigg\{ \delta_r^{f,x+y} + \delta_r^{f,x+y}\hat{V}_r + \delta_r^{f,z}\left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1}\hat{Z}_r - \\ - \hat{Z}_r^\top \left(\delta_r^{\overline{\sigma},x} + \hat{V}_r\delta_r^{\overline{\sigma},y} + \delta_r^{\overline{\sigma},z}\left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1}\left(\hat{Z}_r + \hat{V}_r\left(\delta_r^{\overline{\sigma},x} + \delta_r^{\overline{\sigma},y}\hat{V}_r\right)\right)\right)\Bigg\}\mathrm{d}r, \quad s \in [t,T]. \tag{5.33}$$

In our case $\delta_r^{\overline{\sigma},z}$ is equal to the diagonal $d \times d$-matrix having the value -1 in the first d_1 diagonal entries and 0 everywhere else. Therefore

- $\left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1}$ is a diagonal matrix having $(1+\hat{V}_r)^{-1} = \frac{1}{1+\hat{V}_r}$ in the first d_1 diagonal entries and 1 in the others. So
- $\delta_r^{f,z}\left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1} = \delta_r^{f,z}$, since the first d_1 diagonal entries of $\delta_r^{f,z}$ vanish. Furthermore
- $\delta_r^{\overline{\sigma},z}\left(I_d - \hat{V}_r\delta_r^{\overline{\sigma},z}\right)^{-1}$ is a diagonal matrix having $-(1+\hat{V}_r)^{-1}$ in the first d_1 diagonal entries and 0 everywhere else.

So, we can simplify (5.33):

$$\hat{V}_s = \hat{V}_T - \int_s^T \mathrm{d}B_r^\top \hat{Z}_r - \int_s^T \Bigg(\delta_r^{f,x+y} + \delta_r^{f,x+y}\hat{V}_r + \delta_r^{f,z}\hat{Z}_r - \\ - \hat{Z}_r^\top \left(\delta_r^{\overline{\sigma},x} + \hat{V}_r\delta_r^{\overline{\sigma},y} - \left(1+\hat{V}_r\right)^{-1}\pi_1\left(\hat{Z}_r + \hat{V}_r\left(\delta_r^{\overline{\sigma},x} + \delta_r^{\overline{\sigma},y}\hat{V}_r\right)\right)\right)\Bigg)\mathrm{d}r.$$

Now, use $\delta_r^{\overline{\sigma},y} = \delta_r^{\overline{\sigma},x} = \pi_1(\delta_r^{\overline{\sigma},x})$ to see

- $\delta_r^{\overline{\sigma},x} + \delta_r^{\overline{\sigma},y}\hat{V}_r = \delta_r^{\overline{\sigma},x}(1+\hat{V}_r)$ and
- $\delta_r^{\overline{\sigma},x} + \hat{V}_r\delta_r^{\overline{\sigma},y} - \left(1+\hat{V}_r\right)^{-1}\pi_1\left(\hat{V}_r\left(\delta_r^{\overline{\sigma},x} + \delta_r^{\overline{\sigma},y}\hat{V}_r\right)\right) = \delta_r^{\overline{\sigma},x} + \hat{V}_r\delta_r^{\overline{\sigma},x} - \pi_1\left(\hat{V}_r\delta_r^{\overline{\sigma},x}\right) = \delta_r^{\overline{\sigma},x}$,

s.t. we obtain

$$\hat{V}_s = \hat{V}_T - \int_s^T \mathrm{d}B_r^\top \hat{Z}_r - \int_s^T \Big(\delta_r^{f,x+y} + \delta_r^{f,x+y}\hat{V}_r + \delta_r^{f,z}\hat{Z}_r - \\ - \hat{Z}_r^\top\left(\delta_r^{\overline{\sigma},x}\right) + \hat{Z}_r^\top\left(1+\hat{V}_r\right)^{-1}\pi_1\left(\hat{Z}_r\right)\Big)\,\mathrm{d}r,$$

or

$$\hat{V}_s = \hat{V}_T - \int_s^T \mathrm{d}B_r^\top \hat{Z}_r - \int_s^T \left(\delta_r^{f,x+y}(1+\hat{V}_r) + \left(\delta_r^{f,z} - \left(\delta_r^{\overline{\sigma},x}\right)^\top\right)\hat{Z}_r + \left(1+\hat{V}_r\right)^{-1}\left|\pi_1\left(\hat{Z}_r\right)\right|^2\right)\mathrm{d}r.$$

Now, apply Itô formula to $\ln(1+\hat{V}_s)$:

$$\ln(1+\hat{V}_s) = \ln(1+\hat{V}_T) - \int_s^T \mathrm{d}B_r^\top(1+\hat{V}_r)^{-1}\hat{Z}_r - \\ - \int_s^T \Big(\left(1+\hat{V}_r\right)^{-1}\left(\delta_r^{f,x+y}(1+\hat{V}_r) + \left(\delta_r^{f,z} - \left(\delta_r^{\overline{\sigma},x}\right)^\top\right)\hat{Z}_r + \left(1+\hat{V}_r\right)^{-1}\left|\pi_1\left(\hat{Z}_r\right)\right|^2\right) - \\ - \frac{1}{2}\left(1+\hat{V}_r\right)^{-2}|\hat{Z}_r|^2\Big)\,\mathrm{d}r$$

which after defining $\overline{Z} := (1+\hat{V})^{-1}\hat{Z}$ simplifies to

$$\ln(1+\hat{V}_s) = \ln(1+\hat{V}_T) - \int_s^T \mathrm{d}B_r^\top\overline{Z}_r - \\ - \int_s^T \left(\delta_r^{f,x+y} + \left(\delta_r^{f,z} - \left(\delta_r^{\overline{\sigma},x}\right)^\top\right)\overline{Z}_r + \left|\pi_1\left(\overline{Z}_r\right)\right|^2 - \frac{1}{2}|\overline{Z}_r|^2\right)\mathrm{d}r. \quad (5.34)$$

Let us rewrite this equation as

$$\ln(1+\hat{V}_s) = \ln(1+\hat{V}_T) - \int_s^T \left(\mathrm{d}B_r + \left(\left(\delta_r^{f,z}\right)^\top - \delta_r^{\overline{\sigma},x} + \pi_1(\overline{Z}_r) - \frac{1}{2}\overline{Z}_r\right)\mathrm{d}r\right)^\top \overline{Z}_r - \\ - \int_s^T \delta_r^{f,x+y}\,\mathrm{d}r.$$

Since $\ln(1+\hat{V})$ is a bounded process, $\overline{Z}$ is a BMO process under $\mathbb{Q}$ according to (5.34) and Theorem A.1.11. Furthermore, Z and, thereby, $\delta^{f,z}$ is bounded. Therefore, using some Girsanov measure change we get after exploiting $\delta^{f,x+y} \geq 0$:

$$\ln\left(1+\frac{\mathrm{d}}{\mathrm{d}x}u(t,\check{x},x)\right) = \mathbb{E}_{\mathbb{Q}_1}\left[\ln\left(1+\hat{V}_s\right)\right] \leq \\ \leq \mathbb{E}_{\mathbb{Q}_1}\left[\ln\left(1+\hat{V}_T\right)\right] \leq \ln\left(1+\left\|\frac{\mathrm{d}}{\mathrm{d}x}H\right\|_\infty\right) = \ln(1+L_{H,x}),$$

under some probability measure $\mathbb{Q}_1 \sim \mathbb{Q} \sim \mathbb{P}$. This simplifies to $\frac{\mathrm{d}}{\mathrm{d}x}u(t,\check{x},x) \leq L_{H,x} < L_{\sigma,z}^{-1}$ for almost all $\check{x}, x$. Similarly $\frac{\mathrm{d}}{\mathrm{d}x}u(s,\cdot,\cdot) \leq L_{H,x} < L_{\sigma,z}^{-1}$ a.e. for $s \in [t,T]$, since the same arguments can be applied to the weakly regular Markovian decoupling field $u|_{[s,T]}$.

Uniformly controlling $\frac{\mathrm{d}}{\mathrm{d}x}u$ from below is, however, a bit more challenging and will be based on a rather deep exploitation of the specific structure of the forward-backward system:

Define a Brownian motion with drift via

$$\tilde{B}_s := B_s - B_t + \int_t^s \left(\left(\delta_r^{f,z} \right)^\top - \delta_r^{\overline{\sigma},x} + \pi_1 \left(\overline{Z}_r \right) \right) \mathrm{d}r$$

The BSDE (5.34) can also be rewritten as

$$\ln(1+\hat{V}_s) = \ln(1+\hat{V}_T) - \int_s^T \mathrm{d}\tilde{B}_r^\top \overline{Z}_r - \int_s^T \left(\delta_r^{f,x+y} - \frac{1}{2} \left| \overline{Z}_r \right|^2 \right) \mathrm{d}r.$$

$\tilde{B}$ is a Brownian motion under some probability measure $\tilde{\mathbb{Q}} \sim \mathbb{Q}$ (Theorem A.1.2), so

$$\mathbb{E}_{\tilde{\mathbb{Q}}}[\ln(1+\hat{V}_s)] = \mathbb{E}_{\tilde{\mathbb{Q}}}[\ln(1+\hat{V}_T)] - \mathbb{E}_{\tilde{\mathbb{Q}}} \left[\int_s^T \left(\delta_r^{f,x+y} - \frac{1}{2} \left| \overline{Z}_r \right|^2 \right) \mathrm{d}r \right], \quad s \in [t,T].$$

In order to control $\mathbb{E}_{\tilde{\mathbb{Q}}}[\ln(1+\hat{V}_t)]$ from below we need to control $\mathbb{E}_{\tilde{\mathbb{Q}}} \left[\int_t^T \left(\delta_r^{f,x+y} - \frac{1}{2} \left| \overline{Z}_r \right|^2 \right) \mathrm{d}r \right]$ from above. Remembering the structure of f we have:

$$\delta_r^{f,x+y} = |\pi_1(\theta(r,\varepsilon\check{X}_r))|^2 \left(\frac{U'}{U''} \left(1 - \frac{1}{2} \frac{U^{(3)}U'}{(U'')^2} \right) \right)' (X_r+Y_r) - \frac{1}{2} |\pi_2(Z_r)|^2 \cdot \left(\frac{U^{(3)}}{U''} \right)' (X_r+Y_r). \quad (5.35)$$

Now, define $P := X + Y$. By summing up the forward equation for X and the backward equation (5.32) we obtain the dynamics of P:

$$\begin{aligned} P_s = P_T &- \int_s^T \mathrm{d}B_r^\top \left(\pi_2(Z_r) - \pi_1(\theta(r,\varepsilon\check{X}_r)) \frac{U'}{U''}(P_r) \right) - \\ &- \int_s^T \left(|\pi_1(\theta(r,\varepsilon\check{X}_r))|^2 \frac{U'}{U''} \left(1 - \frac{1}{2} \frac{U^{(3)}U'}{(U'')^2} \right) (P_r) - \frac{1}{2} |\pi_2(Z_r)|^2 \cdot \frac{U^{(3)}}{U''}(P_r) \right) \mathrm{d}r, \end{aligned}$$

Now, define $\varphi := \frac{U'}{U''}$. Clearly, φ is negative. We also know that it is bounded and also bounded away from 0. Remember $\kappa = -\ln(-U'')$, so $\kappa' = -\frac{U^{(3)}}{U''}$. According to the Itô formula the bounded process $\varphi(P)$ has dynamics

$$\begin{aligned} \varphi(P_s) = \varphi(P_T) &- \int_s^T \mathrm{d}B_r^\top \varphi'(P_r) \left(\pi_2(Z_r) - \pi_1(\theta(r,\varepsilon\check{X}_r)) \varphi(P_r) \right) - \\ &- \int_s^T \left\{ \varphi'(P_r) \left(|\pi_1(\theta(r,\varepsilon\check{X}_r))|^2 \varphi \cdot \left(1 + \frac{1}{2}\varphi\kappa' \right)(P_r) + \frac{1}{2} |\pi_2(Z_r)|^2 \kappa'(P_r) \right) + \right. \\ &\left. \qquad + \frac{1}{2}\varphi''(P_r) \left| \pi_2(Z_r) - \pi_1(\theta(r,\varepsilon\check{X}_r))\varphi(P_r) \right|^2 \right\} \mathrm{d}r. \end{aligned}$$

Note

$$\left| \pi_2(Z_r) - \pi_1(\theta(r,\varepsilon\check{X}_r))\varphi(P_r) \right|^2 = |\pi_2(Z_r)|^2 + \left| \pi_1(\theta(r,\varepsilon\check{X}_r))\varphi(P_r) \right|^2,$$

due to orthogonality. So, after regrouping the terms we have

$$\begin{aligned} \varphi(P_r) = \varphi(P_T) &- \int_s^T \mathrm{d}B_r^\top \varphi'(P_r) \left(\pi_2(Z_r) - \pi_1(\theta(r,\varepsilon\check{X}_r)) \varphi(P_r) \right) - \\ - \int_s^T &\left(|\pi_1(\theta(r,\varepsilon\check{X}_r))|^2 \left(\varphi'\varphi \cdot \left(1 + \frac{1}{2}\varphi\kappa' \right) + \frac{1}{2}\varphi''\varphi^2 \right)(P_r) + \frac{1}{2} |\pi_2(Z_r)|^2 \left(\varphi'\kappa' + \varphi'' \right)(P_r) \right) \mathrm{d}r. \end{aligned} \quad (5.36)$$

Now, consider the definition of $\tilde{B}$. Due to the structure of f and $\overline{\sigma}$ we have $\delta_r^{f,z} = \pi_2(Z_r)^\top \kappa'(P_r)$ and $\delta_r^{\overline{\sigma},x} = -\pi_1(\theta(r, \varepsilon\check{X}_r))\varphi'(P_r)$, so

$$\int_s^T \mathrm{d}\tilde{B}_r^\top \varphi'(P_r)\left(\pi_2(Z_r) - \pi_1(\theta(r,\varepsilon\check{X}_r))\varphi(P_r)\right) = \int_s^T \mathrm{d}B_r^\top \varphi'(P_r)\left(\pi_2(Z_r) - \pi_1(\theta(r,\varepsilon\check{X}_r))\varphi(P_r)\right) +$$
$$+ \int_s^T \left(|\pi_2(Z_r)|^2 \varphi'\kappa'(P_r) - |\pi_1(\theta(r,\varepsilon\check{X}_r))|^2 \varphi(\varphi')^2(P_r) - \pi_1(\overline{Z}_r)^\top \pi_1(\theta(r,\varepsilon\check{X}_r))\varphi'\varphi(P_r)\right) \mathrm{d}r,$$

or

$$-\int_s^T \mathrm{d}B_r^\top \varphi'(P_r)\left(\pi_2(Z_r) - \pi_1(\theta(r,\varepsilon\check{X}_r))\varphi(P_r)\right) =$$
$$= -\int_s^T \mathrm{d}\tilde{B}_r^\top \varphi'(P_r)\left(\pi_2(Z_r) - \pi_1(\theta(r,\varepsilon\check{X}_r))\varphi(P_r)\right) -$$
$$-\int_s^T \left(-|\pi_2(Z_r)|^2 \varphi'\kappa'(P_r) + |\pi_1(\theta(r,\varepsilon\check{X}_r))|^2 \varphi(\varphi')^2(P_r) + \pi_1(\overline{Z}_r)^\top \pi_1(\theta(r,\varepsilon\check{X}_r))\varphi'\varphi(P_r)\right) \mathrm{d}r,$$

which together with (5.36) yields

$$\varphi(P_s) = \varphi(P_T) - \int_s^T \mathrm{d}\tilde{B}_r^\top \varphi'(P_r)\left(\pi_2(Z_r) - \pi_1(\theta(r,\varepsilon\check{X}_r))\varphi(P_r)\right) -$$
$$-\int_s^T \Bigg(|\pi_1(\theta(r,\varepsilon\check{X}_r))|^2 \left(\varphi'\varphi\left(1+\varphi'+\frac{1}{2}\varphi\kappa'\right) + \frac{1}{2}\varphi''\varphi^2\right)(P_r) + \frac{1}{2}|\pi_2(Z_r)|^2 \left(\varphi'\kappa' + \varphi'' - 2\kappa'\varphi'\right)(P_r) +$$
$$+ \varphi'\varphi(P_r)\pi_1(\theta(r,\varepsilon\check{X}_r))^\top \overline{Z}_r\Bigg) \mathrm{d}r.$$

We have using the chain rule

- $\varphi' = \frac{U''U''-U'U^{(3)}}{(U'')^2} = 1 + \varphi\kappa'$, which is bounded. Furthermore,
- $\varphi'' = \varphi'\kappa' + \varphi\kappa''$, which is also bounded. Finally
- $\varphi'\kappa' + \varphi'' - 2\kappa'\varphi' = \varphi'\kappa' + \varphi'\kappa' + \varphi\kappa'' - 2\kappa'\varphi' = \varphi\kappa''$.

And so we have after applying conditional expectations

$$\mathbb{E}_{\tilde{\mathbb{Q}}}\left[\varphi(P_t)\right] = \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\varphi(P_T)\right] - \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\int_t^T \left(\alpha_s + \frac{1}{2}\varphi\kappa''(P_s)|\pi_2(Z_s)|^2 + \beta_s^\top \overline{Z}_s\right)\mathrm{d}s\right],$$

with some uniformly bounded progressively measurable processes α, β. This means

$$0 \le \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\int_t^T \frac{-1}{2}\varphi\kappa''(P_s)|\pi_2(Z_s)|^2\,\mathrm{d}s\right] \le \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\int_t^T \beta_s^\top \overline{Z}_s\,\mathrm{d}s\right] + T\|\alpha\|_\infty + 2\|\varphi\|_\infty,$$

Now, note $\delta_r^{f,x+y} = \frac{1}{2}|\pi_2(Z_r)|^2\kappa''(P_r) + \gamma_r$, with some uniformly bounded process γ, according to (5.35). This implies considering $\varphi < 0$:

$$\mathbb{E}_{\tilde{\mathbb{Q}}}\left[\int_t^T \left(\delta_r^{f,x+y} - \frac{1}{2}\left|\overline{Z}_r\right|^2\right)\mathrm{d}r\right] = \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\int_t^T \left(\frac{1}{-\varphi}\frac{-1}{2}\varphi\kappa''(P_r)|\pi_2(Z_r)|^2 + \gamma_r - \frac{1}{2}\left|\overline{Z}_r\right|^2\right)\mathrm{d}r\right] \le$$
$$\le \left\|\frac{1}{-\varphi}\right\|_\infty \left(\mathbb{E}_{\tilde{\mathbb{Q}}}\left[\int_t^T \beta_s^\top \overline{Z}_s\,\mathrm{d}s\right] + T\|\alpha\|_\infty + 2\|\varphi\|_\infty\right) + T\|\gamma\|_\infty - \frac{1}{2}\mathbb{E}_{\tilde{\mathbb{Q}}}\left[\int_t^T \left|\overline{Z}_r\right|^2\mathrm{d}r\right] \le$$

$$\leq \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\int_t^T \frac{1}{2}\left|\left\|\frac{1}{-\varphi}\right\|_\infty \beta_s\right|^2 \mathrm{d}s\right] + \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\int_t^T \frac{1}{2}|\overline{Z}_s|^2 \,\mathrm{d}s\right] +$$

$$+\left\|\frac{1}{-\varphi}\right\|_\infty (T\|\alpha\|_\infty + 2\|\varphi\|_\infty) + T\|\gamma\|_\infty - \frac{1}{2}\mathbb{E}_{\tilde{\mathbb{Q}}}\left[\int_t^T \left|\overline{Z}_r\right|^2 \mathrm{d}r\right] \leq$$

$$\leq \frac{1}{2}\left\|\frac{1}{-\varphi}\right\|_\infty^2 T\|\beta\|_\infty^2 + \left\|\frac{1}{-\varphi}\right\|_\infty (T\|\alpha\|_\infty + 2\|\varphi\|_\infty) + T\|\gamma\|_\infty < \infty.$$

This is a uniform bound we were looking for! This means that

$$\mathbb{E}_{\tilde{\mathbb{Q}}}[\ln(1+\hat{V}_s)] = \ln\left(1 + \frac{\mathrm{d}}{\mathrm{d}x}u(t,\check{x},x)\right) \geq -C$$

for a.a. $(\check{x},x)$, where $C > 0$ does not depend on t, or u or ε which immediately implies that $\frac{\mathrm{d}}{\mathrm{d}x}u(t,\cdot,\cdot)$ is uniformly bounded away from -1. The same bound works for $\frac{\mathrm{d}}{\mathrm{d}x}u(s,\cdot,\cdot)$, $s \in [t,T]$. ✓

And so we have controlled $\frac{\mathrm{d}}{\mathrm{d}x}u(s,\cdot,\cdot)$ from both sides such that its modulus is bounded uniformly away from 1 (independently of t, u, ε, as long as ε is sufficiently small for the problem to satisfy MLLC). This shows that Theorem 5.3.12 is applicable and we have $I^M_{\max} = [0,T]$ for some $\varepsilon > 0$. In particular, the FBSDE given by (5.31) and (5.32) for the interval $[0,T]$ has a solution $\check{X}, Y, Z$ s.t. $\|Z\|_\infty < \infty$ for any initial value $(\check{x},x) \in \mathbb{R}^N \times \mathbb{R}$.
Furthermore, this solution is unique: Assume there is another such triple $(\check{X}', Y', Z')$. Then due to boundedness of Z' and the dynamics of Y', the process Y' must be bounded as well. At the same time the dynamics of X' imply that it will at least satisfy $\sup_{s\in[0,T]} \mathbb{E}_{\mathbb{Q}\,0,\infty}[(X'_s)^2] < \infty$. Similar properties hold true for X and Y, so Lemma 5.3.8 is applicable and the triples must coincide. □

Remark 5.3.14. Using the Itô formula it is straightforward to verify that the processes X, Y, Z from Theorem 5.3.13 satisfy:

$$U'(X_t + Y_t) = U'(X_0 + Y_0) + \int_0^t U'(X_s + Y_s)\alpha_s^\top \,\mathrm{d}W_s \qquad \text{a.s.} \qquad \forall t \in [0,T],$$

where $\alpha_s := \frac{U''}{U'}(X_s + Y_s)\pi_2(Z_s) - \pi_1(\theta_s)$, $s \in [0,T]$. This implies that $t \mapsto U'(X_t + Y_t)$ describes a uniformly integrable martingale due to boundedness of $\frac{U''}{U'}$, Z and θ.

Appendix A

Appendix

A.1 BMO - processes and their properties

In the following, let $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a complete filtered probability space, such that the filtration satisfies the usual hypotheses. Assume furthermore that there exists a d-dimensional Brownian motion W on $[0,T]$, which is progressive w.r.t. $(\mathcal{F}_t)_{t\in[0,T]}$, independent of $\mathcal{F}_0$ and such that $\mathcal{F}_t = \sigma(\mathcal{F}_0, \mathcal{F}_t^W)$, where $\mathcal{F}^W$ is the natural filtration generated by W and $\mathcal{F}_0$ contains every null set.

For a probability measure $\mathbb{Q} \sim \mathbb{P}$ and any $q > 0$ and $m \in \mathbb{N}$ define $\mathcal{H}^q(\mathbb{R}^m, \mathbb{Q})$ as the space of all progressive processes $(Z_t)_{t\in[0,T]}$ with values in $\mathbb{R}^m$ normed by

$$\|Z\|_{\mathcal{H}^q} := \left(\mathbb{E}_{\mathbb{Q}}\left[\left(\int_0^T |Z_s|^2 \, \mathrm{d}s\right)^{\frac{q}{2}}\right]\right)^{\frac{1}{q}} < \infty.$$

Definition A.1.1. Let $\mathbb{Q} \sim \mathbb{P}$ be an equivalent probability measure and define

$$BMO(\mathbb{Q}) := \left\{ Z \colon [0,T] \times \Omega \,\middle|\, Z \text{ is progressively measurable and vector-valued s.t.} \right.$$
$$\left. \exists C \geq 0 \, \forall t \in [0,T] : \mathbb{E}_{\mathbb{Q}}\left[\int_t^T |Z_s|^2 \, \mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \leq C \text{ a.s.} \right\}.$$

By vector-valued we mean that Z should assume values in some normed vector space.
The smallest constant C such that the above bound holds is denoted by $\check{C} =: \|Z\|^2_{BMO(\mathbb{Q})}$. For processes $Z \notin BMO(\mathbb{Q})$ we define $\|Z\|_{BMO(\mathbb{Q})} := \infty$.
Also, we refer to $\|Z\|_{BMO(\mathbb{Q})}$ as the $BMO(\mathbb{Q})$ - norm of Z.

We will sometimes refer to such $Z \in BMO(\mathbb{Q})$ as *BMO - processes*. If $\mathbb{Q}$ is not otherwise specified, this term implies $Z \in BMO(\mathbb{P})$. We might also sometimes use the term $BMO(\mathbb{Q})$ - process in this context to specify the probability measure.

Furthermore, we call a martingale M a BMO-martingale if

$$M_t = M_0 + \int_0^t Z_s \, \mathrm{d}W_s =: M_0 + (Z \bullet W)_t, \quad t \in [0,T]$$

with some $\mathbb{R}^{1\times d}$-valued $Z \in BMO(\mathbb{P})$ and an $M_0 \in L^2(\mathcal{F}_0)$.

Also, if a progressive process Z is only defined on a subinterval of $[0,T]$, the statement $Z \in BMO(\mathbb{Q})$ means that its natural extension to $[0,T]$, obtained by setting it to 0 everywhere outside its initial domain, is in $BMO(\mathbb{Q})$.

Theorem A.1.2 (Theorem 2.3. in [Kaz94])**.** *Let $\mu \in BMO(\mathbb{P})$ be $\mathbb{R}^{1\times d}$-valued, then*

$$\mathbb{Q}^\mu := \mathcal{E}(\mu \bullet W)_T \cdot \mathbb{P}$$

is a probability measure.

Lemma A.1.3. *For a probability measure $\mathbb{Q} \sim \mathbb{P}$ let $Z \in BMO(\mathbb{Q})$ be $\mathbb{R}^m$-valued. Then $Z \in \mathcal{H}^{2n}(\mathbb{R}^m, \mathbb{Q})$ for all $n \in \mathbb{N}$ and*

$$\left(\mathbb{E}_\mathbb{Q}\left[\left(\int_t^T |Z_s|^2 \, \mathrm{d}s\right)^n \,\middle|\, \mathcal{F}_t\right]\right)^{\frac{1}{2n}} \leq \sqrt[2n]{n!}\, \|Z\|_{BMO(\mathbb{Q})}$$

a.s. for all $t \in [0,T]$. In particular, $\|Z\|_{\mathcal{H}^{2n}(\mathbb{R}^m,\mathbb{Q})} \leq \sqrt[2n]{n!}\, \|Z\|_{BMO(\mathbb{Q})}$.

Proof. Define $A_t := \int_0^t |Z_s|^2 \, \mathrm{d}s$ for all $t \in [0,T]$. A is progressive, continuous, non-decreasing and satisfies

$$\mathbb{E}_\mathbb{Q}[A_T - A_t | \mathcal{F}_t] \leq \|Z\|^2_{BMO(\mathbb{Q})}$$

for all $t \in [0,T]$. Therefore, using energy inequalities (consult [Kik92] for instance) we have

$$\mathbb{E}_\mathbb{Q}[(A_T - A_t)^n | \mathcal{F}_t] \leq n! \left(\|Z\|^2_{BMO(\mathbb{Q})}\right)^n,$$

which implies the assertion. □

Lemma A.1.4. *For all $K > 0$ there exists a constant $C > 0$, which is increasing in K, such that*

$$\mathbb{E}_\mathbb{Q}\left[\exp\left(\int_t^T |Z_s| \, \mathrm{d}s\right) \middle| \mathcal{F}_t\right] \leq C \quad \text{a.s. for all } t \in [0,T],$$

for all probability measures $\mathbb{Q} \sim \mathbb{P}$ and all $Z \in BMO(\mathbb{Q})$ such that $\|Z\|_{BMO(\mathbb{Q})} \leq K$.

Proof. We apply Lemma A.1.3 and the Cauchy-Schwarz inequality:

$$\begin{aligned}
\mathbb{E}_\mathbb{Q}\left[\exp\left(\int_t^T |Z_s| \, \mathrm{d}s\right) \middle| \mathcal{F}_t\right] &= \mathbb{E}_\mathbb{Q}\left[\sum_{k=0}^\infty \frac{1}{k!}\left(\int_t^T |Z_s| \, \mathrm{d}s\right)^k \middle| \mathcal{F}_t\right] \\
&\leq \sum_{k=0}^\infty \frac{1}{k!}\mathbb{E}_\mathbb{Q}\left[\left(\int_t^T |Z_s| \, \mathrm{d}s\right)^k \middle| \mathcal{F}_t\right] \leq \sum_{k=0}^\infty \frac{1}{k!}\mathbb{E}_\mathbb{Q}\left[\left((T-t)\int_t^T |Z_s|^2 \, \mathrm{d}s\right)^{\frac{k}{2}} \middle| \mathcal{F}_t\right] \\
&\leq \sum_{k=0}^\infty \frac{1}{k!}\left(\mathbb{E}_\mathbb{Q}\left[\left(T\int_t^T |Z_s|^2 \, \mathrm{d}s\right)^k \middle| \mathcal{F}_t\right]\right)^{\frac{1}{2}} \leq \sum_{k=0}^\infty \frac{T^{\frac{k}{2}}}{k!}\left(k!\left(\|Z\|_{BMO(\mathbb{Q})}\right)^{2k}\right)^{\frac{1}{2}} \\
&\leq \sum_{k=0}^\infty \frac{T^{\frac{k}{2}}}{\sqrt{k!}} K^k =: C < \infty,
\end{aligned}$$

since

$$\left(\frac{T^{\frac{k+1}{2}}}{\sqrt{(k+1)!}} K^{k+1}\right) \cdot \left(\frac{T^{\frac{k}{2}}}{\sqrt{k!}} K^k\right)^{-1} = \frac{T^{\frac{1}{2}}}{\sqrt{k+1}} K \to 0, \quad k \to \infty,$$

so the series converges absolutely and is monotonically increasing in K. □

Theorem A.1.5. *Let $Z \in BMO(\mathbb{Q})$ be $\mathbb{R}^{1\times d}$ - valued and such that $\|Z\|_{BMO(\mathbb{Q})} < \frac{1}{4}$. Then for any $t \in [0,T]$:*

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left(\left|\int_t^T Z_s \,\mathrm{d}\tilde{W}_s\right|\right)\middle|\mathcal{F}_t\right] \leq \frac{1}{1-4\|Z\|_{BMO(\mathbb{Q})}} \quad a.s.,$$

where $\tilde{W}$ is a d - dimensional Brownian motion under $\mathbb{Q}$.

Proof. Theorem 2.1. in [Kaz94]. □

Theorem A.1.6. *Let $\mu \in BMO(\mathbb{P})$ be $\mathbb{R}^{1\times d}$-valued. Define the probability measure $\mathbb{Q}^{\mu} := \mathcal{E}(\mu \bullet W)_T \cdot \mathbb{P}$. Then for all progressively measurable processes Z:*

$$\|Z\|_{BMO(\mathbb{Q}^{\mu})} \leq K_1\|Z\|_{BMO(\mathbb{P})} \quad \text{and} \quad \|Z\|_{BMO(\mathbb{P})} \leq K_2\|Z\|_{BMO(\mathbb{Q}^{\mu})}$$

with some real constants $K_1, K_2 > 0$ depending only on $\|\mu\|_{BMO(\mathbb{P})}$ and montonically increasing in this value.

Proof. See proof of Theorem 3.6. in [Kaz94] and also Theorem 2.4. in [Kaz94]. □

As an application let us proof the following statement:

Lemma A.1.7. *For some $N \in \mathbb{N}$ let Y be an $\mathbb{R}^{1\times N}$-valued progressively measurable bounded process on $[0,T]$ with dynamics given by*

$$Y_s = Y_T - \int_s^T \mathrm{d}W_r^{\top} Z_r - \int_s^T \left(\alpha_r + Y_r\left(\delta_r I_N + \beta_r\right) + \sum_{i=1}^{d} Z_r^i \gamma_r^i + \mu_r^{\top} Z_r\right) \mathrm{d}r, \quad s \in [0,T], \tag{A.1}$$

where

- *Y_T is $\mathbb{R}^{1\times N}$-valued, $\mathcal{F}_T$-measurable and bounded,*
- *Z is some $\mathbb{R}^{d\times N}$-valued progressively measurable process such that Z can also be interpreted as a vector $(Z^i)_{i=1,\ldots,d}$ of $\mathbb{R}^{1\times N}$-valued processes Z^i, $i = 1,\ldots,d$ with $\int_0^T |Z_s^i|^2 \,\mathrm{d}s < \infty$ a.s.,*
- *α is an $\mathbb{R}^{1\times N}$-valued $BMO(\mathbb{P})$-process,*
- *δ is some non-negative progressively measurable process with $\int_0^T \delta_s \,\mathrm{d}s < \infty$ a.s.,*
- *$I_N \in \mathbb{R}^{N\times N}$ is the identity matrix,*
- *β is an $\mathbb{R}^{N\times N}$-valued $BMO(\mathbb{P})$-process,*
- *γ^i, $i = 1,\ldots,d$, are progressively measurable and bounded $\mathbb{R}^{N\times N}$-valued processes,*
- *μ is an $\mathbb{R}^d$-valued $BMO(\mathbb{P})$-process.*

Then Y is bounded by

$$\|Y\|_{\infty} \leq C_1 \cdot \|Y_T\|_{\infty} + C_2 \cdot \|\alpha\|_{BMO(\mathbb{P})},$$

with constants $C_1, C_2 \in [0,\infty)$ which depend only on T, $\|\beta\|_{BMO(\mathbb{P})}$, $\|\mu\|_{BMO(\mathbb{P})}$ and $\|\gamma^{(i)}\|_{\infty}$, $i = 1,\ldots,d$, and are monotonically increasing in these values.

Proof. In order to get rid of the term $\mu_r^\top Z_r$ we define a Brownian motion with drift on $[0,T]$ via

$$\tilde{W}_s := W_s + \int_0^s \mu_r \,\mathrm{d}r, \quad s \in [0,T]$$

Using a standard Girsanov measure change $\tilde{W}$ is a Brownian motion w.r.t. to some equivalent probability measure $\mathbb{Q}$. Furthermore, using (A.1) the process Y has dynamics

$$Y_s = Y_T - \int_s^T \mathrm{d}\tilde{W}_r^\top Z_r - \int_s^T \left(\alpha_r + Y_r\left(\delta_r I_N + \beta_r\right) + \sum_{i=1}^d Z_r^i \gamma_r^i\right) \mathrm{d}r, \quad s \in [0,T].$$

Now, choose a $t \in [0,T]$. We want to control Y_t. For that purpose define

$$\Gamma_s := \exp\left(-\int_t^s \left(\delta_r I_N + \beta_r\right) \mathrm{d}r - \int_t^s \sum_{i=1}^d \mathrm{d}\tilde{W}_r^i \gamma_r^i - \frac{1}{2}\int_{t_1}^s \sum_{i=1}^d \gamma_r^i \gamma_r^i \,\mathrm{d}r\right), \quad s \in [t,T].$$

According to the Itô formula Γ has dynamics

$$\Gamma_s = \Gamma_T + \int_s^T \sum_{i=1}^d \mathrm{d}\tilde{W}_r^i \gamma_r^i \Gamma_r + \int_s^T \left(\delta_r I_N + \beta_r\right)\Gamma_r \,\mathrm{d}r,$$

$s \in [t,T]$. Now, apply the Itô formula to $Y_s\Gamma_s$:

$$\begin{aligned} Y_s\Gamma_s = Y_T\Gamma_T &- \int_s^T \sum_{i=1}^d \mathrm{d}\tilde{W}_r^i \left(Z_r^i\Gamma_r - Y_r\gamma_r^i\Gamma_r\right) - \\ &- \int_s^T \left\{\left(\alpha_r + Y_r\left(\delta_r I_N + \beta_r\right) + \sum_{i=1}^d Z_r^i\gamma_r^i\right)\Gamma_r - Y_r\left(\delta_r I_N + \beta_r\right)\Gamma_r - \sum_{i=1}^d Z_r^i\gamma_r^i\Gamma_r\right\} \mathrm{d}r. \end{aligned}$$

A few terms cancel out and we end up with

$$Y_s\Gamma_s = Y_T\Gamma_T - \int_s^T \sum_{i=1}^d \mathrm{d}\tilde{W}_r^i \left(Z_r^i\Gamma_r - Y_r\gamma_r^i\Gamma_r\right) - \int_s^T \alpha_r\Gamma_r \,\mathrm{d}r. \tag{A.2}$$

We now want to control $\sup_{s\in[t,T]}|\Gamma_s|$: Observe that due to $\delta \geq 0$ we have for all $p \geq 1$

$$\begin{aligned} \mathbb{E}_\mathbb{Q}\left[\sup_{s\in[t,T]}|\Gamma_s|^p \,\middle|\, \mathcal{F}_t\right] &= \mathbb{E}_\mathbb{Q}\left[\sup_{s\in[t,T]}\left|\exp\left(-\int_t^s \delta_r \,\mathrm{d}r\right)\cdot\right.\right. \\ &\qquad \left.\left.\cdot\exp\left(-\int_t^s \beta_r \,\mathrm{d}r - \int_t^s \sum_{i=1}^d \mathrm{d}\tilde{W}_r^i\gamma_r^i - \frac{1}{2}\int_t^s \sum_{i=1}^d \gamma_r^i\gamma_r^i \,\mathrm{d}r\right)\right|^p \,\middle|\, \mathcal{F}_t\right] \leq \\ &\leq \mathbb{E}_\mathbb{Q}\left[\sup_{s\in[t,T]}\left|\exp\left(\left|\int_t^s \beta_r \,\mathrm{d}r\right| + \left|\int_t^s \sum_{i=1}^d \mathrm{d}\tilde{W}_r^i\gamma_r^i\right| + \frac{1}{2}\left|\int_t^s \sum_{i=1}^d \gamma_r^i\gamma_r^i \,\mathrm{d}r\right|\right)\right|^p \,\middle|\, \mathcal{F}_t\right] \leq \\ &\leq \mathbb{E}_\mathbb{Q}\left[\sup_{s\in[t,T]}\exp\left(p\int_t^s |\beta_r| \,\mathrm{d}r + \frac{p}{2}T\|\gamma\|_\infty^2\right)\cdot\sup_{s\in[t,T]}\exp\left(p\left|\int_t^s \sum_{i=1}^d \mathrm{d}\tilde{W}_r^i\gamma_r^i\right|\right) \,\middle|\, \mathcal{F}_t\right], \end{aligned}$$

which using Cauchy-Schwarz inequality can be further controlled by

$$\left(\mathbb{E}_\mathbb{Q}\left[\exp\left(\int_t^T 2p|\beta_r| \,\mathrm{d}r + pT\|\gamma\|_\infty^2\right) \,\middle|\, \mathcal{F}_t\right]\right)^{\frac{1}{2}} \left(\mathbb{E}_\mathbb{Q}\left[\sup_{s\in[t,T]}\exp\left(2p\left|\int_t^s \sum_{i=1}^d \mathrm{d}\tilde{W}_r^i\gamma_r^i\right|\right) \,\middle|\, \mathcal{F}_t\right]\right)^{\frac{1}{2}}.$$

According to Lemma A.1.4 the first of the two factors above can be controlled by a finite constant, which depends only on p, $\|\beta\|_{BMO(\mathbb{Q})}$, $\|\gamma\|_\infty$ and T and is monotonically increasing in these values. Also, note that $\|\beta\|_{BMO(\mathbb{Q})}$ can be controlled by $\|\beta\|_{BMO(\mathbb{P})}$ and $\|\mu\|_{BMO(\mathbb{P})}$ according to Theorem A.1.6.
The second factor can be estimated using Doob's martingale inequality:

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left(2p\sup_{s\in[t,T]}\left|\int_t^s\sum_{i=1}^d \mathrm{d}\tilde{W}_r^i\gamma_r^i\right|\right)\middle|\mathcal{F}_t\right]\le\sum_{k=0}^\infty\frac{1}{k!}\mathbb{E}_{\mathbb{Q}}\left[\left(\sup_{s\in[t,T]}\left|\int_t^s\sum_{i=1}^d \mathrm{d}\tilde{W}_r^i\left(2p\gamma_r^i\right)\right|\right)^k\middle|\mathcal{F}_t\right]\le$$

$$\le 1+\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[t,T]}\left|\int_t^s\sum_{i=1}^d \mathrm{d}\tilde{W}_r^i\left(2p\gamma_r^i\right)\right|\middle|\mathcal{F}_t\right]+\sum_{k=2}^\infty\frac{1}{k!}\left(\frac{k}{k-1}\right)^k\mathbb{E}_{\mathbb{Q}}\left[\left|\int_t^T\sum_{i=1}^d \mathrm{d}\tilde{W}_r^i\left(2p\gamma_r^i\right)\right|^k\middle|\mathcal{F}_t\right].$$

Using Cauchy-Schwarz inequality and Doob's martingale inequality again, the above value can be controlled by

$$1+2\left(\mathbb{E}_{\mathbb{Q}}\left[\left|\int_t^T\sum_{i=1}^d \mathrm{d}\tilde{W}_r^i\left(2p\gamma_r^i\right)\right|^2\middle|\mathcal{F}_t\right]\right)^{\frac{1}{2}}+\sum_{k=2}^\infty\frac{1}{k!}4\mathbb{E}_{\mathbb{Q}}\left[\left|\int_t^T\sum_{i=1}^d \mathrm{d}\tilde{W}_r^i\left(2p\gamma_r^i\right)\right|^k\middle|\mathcal{F}_t\right]\le$$

$$\le 10\mathbb{E}_{\mathbb{Q}}\left[\exp\left(2p\left|\int_t^T\sum_{i=1}^d \mathrm{d}\tilde{W}_r^i\gamma_r^i\right|\right)\middle|\mathcal{F}_t\right].$$

This value is bounded by a finite constant, which depends only on p, T and $\|\gamma\|_\infty$ and is monotonically increasing in these values: For instance use Theorem A.1.5 by applying it to finitely many sufficiently small subintervals of $[t,T]$ such that $2p\|\gamma\|_\infty$ multiplied by the square root of the size of every subinterval is smaller $\frac{1}{5}$. Also, use the triangle inequality and the tower property after splitting up the stochastic integral. ✓

One implication of the above control for $\sup_{s\in[t,T]}|\Gamma_s|$ is that the stochastic integral in (A.2) represents a uniformly integrable martingale w.r.t. $\mathbb{Q}$ since

$$\int_t^s\sum_{i=1}^d \mathrm{d}\tilde{W}_r^i\left(Z_r^i\Gamma_r-Y_r\gamma_r^i\Gamma_r\right)=Y_s\Gamma_s-Y_t\Gamma_t-\int_t^s\alpha_r\Gamma_r\,\mathrm{d}r\quad\text{a.s. for all } s\in[t,T]$$

and, therefore, using triangle inequality, Cauchy-Schwarz inequality and simple estimates

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[t,T]}\left|\int_t^s\sum_{i=1}^d \mathrm{d}\tilde{W}_r^i\left(Z_r^i\Gamma_r-Y_r\gamma_r^i|\Gamma_r|\right)\right|\right]\le 2\|Y\|_\infty\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[t,T]}|\Gamma_s|\right]+\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[t,T]}\left|\int_t^s\alpha_r\Gamma_r\,\mathrm{d}r\right|\right]\le$$

$$\le 2\|Y\|_\infty\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[t,T]}|\Gamma_s|\right]+\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[t,T]}|\Gamma_s|\int_t^T|\alpha_r|\,\mathrm{d}r\right]\le$$

$$\le 2\|Y\|_\infty\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[t,T]}|\Gamma_s|\right]+\left(\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[t,T]}|\Gamma_s|^2\right]\mathbb{E}_{\mathbb{Q}}\left[T\int_t^T|\alpha|_r^2\,\mathrm{d}r\right]\right)^{\frac{1}{2}},$$

which is finite due to $\alpha\in BMO(\mathbb{P})$ and Theorem A.1.6. ✓

We can finally estimate using (A.2) and Cauchy-Schwarz inequality:

$$|Y_t|=|\mathbb{E}_{\mathbb{Q}}\left[Y_t\Gamma_t|\mathcal{F}_t\right]|=\left|\mathbb{E}_{\mathbb{Q}}\left[Y_T\Gamma_T|\mathcal{F}_t\right]-\mathbb{E}_{\mathbb{Q}}\left[\int_t^T\alpha_r\Gamma_r\,\mathrm{d}r\middle|\mathcal{F}_t\right]\right|\le$$

$$\leq \|Y_T\|_\infty \mathbb{E}_\mathbb{Q}\left[|\Gamma_T||\mathcal{F}_t\right] + \left(\mathbb{E}_\mathbb{Q}\left[\sup_{s\in[t,T]}|\Gamma_s|^2\middle|\mathcal{F}_t\right]\right)^{\frac{1}{2}}\left(\mathbb{E}_\mathbb{Q}\left[T\int_t^T|\alpha_r|^2\,\mathrm{d}r\middle|\mathcal{F}_t\right]\right)^{\frac{1}{2}} \leq$$

$$\leq \|Y_T\|_\infty\sqrt{\mathbb{E}_\mathbb{Q}\left[|\Gamma_T|^2|\mathcal{F}_t\right]} + \sqrt{T}\|\alpha\|_{BMO(\mathbb{Q})}\left(\mathbb{E}_\mathbb{Q}\left[\sup_{s\in[t,T]}|\Gamma_s|^2\middle|\mathcal{F}_t\right]\right)^{\frac{1}{2}} \leq$$

$$\leq \|Y_T\|_\infty\sqrt{\mathbb{E}_\mathbb{Q}\left[|\Gamma_T|^2|\mathcal{F}_t\right]} + K_1\|\alpha\|_{BMO(\mathbb{P})}\left(\mathbb{E}_\mathbb{Q}\left[\sup_{s\in[t,T]}|\Gamma_s|^2\middle|\mathcal{F}_t\right]\right)^{\frac{1}{2}},$$

where we again used Theorem A.1.6. K_1 depends only on $\|\mu\|_{BMO(\mathbb{P})}$ and T. □

Lemma A.1.8. *For all $K > 0$ there exist real constants $r, C > 0$ such that*

$$\mathbb{E}\left[\exp\left(r\int_0^T|Z_t|^2\,\mathrm{d}t\right)\right] \leq C$$

for all $Z \in BMO(\mathbb{P})$ such that $\|Z\|_{BMO(\mathbb{P})} \leq K$.

Proof. Set $r := \frac{1}{2K^2}$ and $\tilde{Z} := \sqrt{r}Z$. Then according to Theorem 2.2. of [Kaz94]

$$\mathbb{E}\left[\exp\left(r\int_0^T|Z_t|^2\,\mathrm{d}t\right)\right] = \mathbb{E}\left[\exp\left(\int_0^T|\tilde{Z}_t|^2\,\mathrm{d}t\right)\right]$$
$$\leq \frac{1}{1-\|\tilde{Z}\|^2_{BMO(\mathbb{P})}} = \frac{1}{1-r\|Z\|^2_{BMO(\mathbb{P})}} \leq 2 =: C.$$

□

Theorem A.1.9. *For all $K > 0$ there exist $p > 1$ and $C > 0$ such that*

$$\mathbb{E}[\mathcal{E}(M)_T^p] = \mathbb{E}\left[\exp\left(\int_0^T Z_s\,\mathrm{d}W_s - \frac{1}{2}\int_0^T|Z_s|^2\,\mathrm{d}s\right)^p\right] \leq C$$

for all BMO-martingales $M = Z \bullet W$ satisfying $\|Z\|_{BMO(\mathbb{P})} \leq K$.

Proof. Define a continuous function $\Phi : (1,\infty) \to (0,\infty)$ via

$$\Phi(p) := \left(1 + \frac{1}{p^2}\ln\frac{2p-1}{2(p-1)}\right)^{\frac{1}{2}} - 1, \quad \text{for } p > 1.$$

According to the proof of Theorem 3.1. of [Kaz94] the inequality $\|Z\|_{BMO(\mathbb{P})} < \Phi(p)$ would imply

$$\mathbb{E}[\mathcal{E}(M)_T^p] \leq \frac{2}{1-2(p-1)(2p-1)^{-1}\exp(p^2\|Z\|_{BMO(\mathbb{P})}(2+\|Z\|_{BMO(\mathbb{P})}))} < \infty.$$

Note that $\|Z\|_{BMO(\mathbb{P})} < \Phi(p)$ also implies

$$\|Z\|_{BMO(\mathbb{P})}(2+\|Z\|_{BMO(\mathbb{P})}) = (\|Z\|_{BMO(\mathbb{P})}+1)^2 - 1 < \frac{1}{p^2}\ln\frac{2p-1}{2(p-1)}.$$

Since $\lim_{p\downarrow 1}\Phi(p) = \infty$, we can choose $p > 1$ s.t. $K < \Phi(p)$, which proves the assertion. □

Lemma A.1.10. *Let Y, Z, ψ, φ be some progressively measurable processes on $[0,T]$ such that*

- *Y is real-valued with bounded Y_T,*
- *Z is $\mathbb{R}^{1\times d}$-valued and in $BMO(\mathbb{P})$,*
- *φ is $\mathbb{R}^d$-valued and in $BMO(\mathbb{P})$,*
- *ψ is real-valued and such that $\sqrt{|\psi|}$ is in $BMO(\mathbb{P})$ and*

$$Y_t = Y_T - \int_t^T (\psi_s + Z_s\varphi_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s$$

holds a.s. for every $t \in [0,T]$.
Then $\|Y\|_\infty \le \|Y_T\|_\infty + \left\|\sqrt{|\psi|}\right\|^2_{BMO(\mathbb{Q})} < \infty$, where $\mathbb{Q} := \mathcal{E}(\varphi \bullet W)_T \cdot \mathbb{P}$. Furthermore, if $\varphi = 0$ it is enough if Z is only progressively measurable and in $L^2([0,T]\times\Omega)$ instead of $BMO(\mathbb{P})$.
Also, note that $\psi \in BMO(\mathbb{P})$ suffices for $\sqrt{|\psi|} \in BMO(\mathbb{P})$ using Cauchy-Schwarz inequality.

Proof. Write

$$Y_t = Y_T - \int_t^T \psi_s\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s^\varphi$$

with $W_s^\varphi := W_s + \int_0^t \varphi_s\,\mathrm{d}s$. Now, perform a change of measure to turn W^φ into a Brownian motion using Girsanov's theorem and apply conditional expectations w.r.t. the new measure and $\mathcal{F}_t$ on both sides. Also, use $\sqrt{|\psi|} \in BMO(\mathbb{P})$ and Theorem A.1.6. □

The following theorem is an extension of a result from [BE09].

Theorem A.1.11. *Let Y, Z, X, ψ, φ be some progressively measurable processes on $[0,T]$ such that*

- *Y is real-valued and bounded,*
- *Z is $\mathbb{R}^{1\times d}$ - valued and s.t. $\int_0^T |Z_s|^2 < \infty$ a.s.,*
- *ψ, φ are real-valued and in $BMO(\mathbb{P})$,*
- *X is real-valued and satisfies $X \le \psi^2 + |Z|\varphi + C|Z|^2$ a.e. with some constant $C \in (0,\infty)$.*

Assume furthermore

$$Y_t = Y_T + \int_t^T X_s\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s \quad a.s., \qquad t \in [0,T].$$

Then we have $\|Z\|_{BMO(\mathbb{P})} \le K < \infty$ for some constant K, which depends only on $\|Y\|_\infty$, C, $\|\varphi\|_{BMO(\mathbb{P})}$, $\|\psi\|_{BMO(\mathbb{P})}$ and is monotonically increasing in theses values.

Proof. Clearly, we have

$$X \le \psi^2 + |Z|\varphi + C|Z|^2 \le (\psi^2 + \frac{1}{2}\varphi^2) + (C + \frac{1}{2})|Z|^2.$$

Define $\tilde{\psi} := \sqrt{\psi^2 + \frac{1}{2}\varphi^2} \in BMO(\mathbb{P})$, $\tilde{C} := C + \frac{1}{2}$, and write

$$Y_t = Y_0 - \int_0^t X_s\,\mathrm{d}s + \int_0^t Z_s\,\mathrm{d}W_s.$$

Let $\beta \in \mathbb{R}$ be some constant specified later. Using Itô's formula we have

$$\exp(\beta Y_t) = \exp(\beta Y_0) - \int_0^t \beta\exp(\beta Y_s)X_s\,\mathrm{d}s + \int_0^t \beta\exp(\beta Y_s)Z_s\,\mathrm{d}W_s + \frac{\beta^2}{2}\int_0^t \exp(\beta Y_s)|Z_s|^2\,\mathrm{d}s$$

and so for every stopping time $\tau \in [t,T]$ we can write

$$\exp(\beta Y_t) = \exp(\beta Y_\tau) + \int_t^\tau \beta \exp(\beta Y_s) X_s \,\mathrm{d}s - \int_t^\tau \beta \exp(\beta Y_s) Z_s \,\mathrm{d}W_s - \frac{\beta^2}{2} \int_t^\tau \exp(\beta Y_s)|Z_s|^2 \,\mathrm{d}s,$$

which can be rearranged to

$$\beta \int_t^\tau \exp(\beta Y_s) \left(\frac{\beta}{2}|Z_s|^2 - X_s\right) \mathrm{d}s = \exp(\beta Y_\tau) - \exp(\beta Y_t) - \int_t \tau\beta \exp(\beta Y_s) Z_s \,\mathrm{d}W_s,$$

or to

$$\begin{aligned}\beta \int_t^\tau & \exp(\beta Y_s) \left(\frac{\beta}{2}|Z_s|^2 + \tilde{\psi}_s^2 - X_s\right) \mathrm{d}s \\ &= \exp(\beta Y_\tau) - \exp(\beta Y_t) + \beta \int_t^\tau \exp(\beta Y_s)\tilde{\psi}_s^2 \,\mathrm{d}s - \int_t^\tau \beta \exp(\beta Y_s) Z_s \,\mathrm{d}W_s.\end{aligned}$$

Setting $\beta := 2\tilde{C} + 2 = 2C + 3$, we have

$$|Z_s|^2 \leq \frac{\beta}{2}|Z_s|^2 + \tilde{\psi}_s^2 - X_s.$$

Now, choose a localizing sequence of stopping times $\tau_n \in [t,T]$, $n \in \mathbb{N}$, s.t. $\mathbb{E}\left[\int_t^{\tau_n} |Z_s|^2 \,\mathrm{d}s\right] < \infty$ for every $n \in \mathbb{N}$ while $\tau_n \uparrow T$ for $n \to \infty$. Applying conditional expectations we have

$$\begin{aligned}\mathbb{E}\left[\beta \int_t^{\tau_n} \exp(\beta Y_s)|Z_s|^2 \,\mathrm{d}s \middle| \mathcal{F}_t\right] &\leq \mathbb{E}\left[\beta \int_t^{\tau_n} \exp(\beta Y_s) \left(\frac{\beta}{2}|Z_s|^2 + \tilde{\psi}_s^2 - X_s\right) \mathrm{d}s\right] \\ &\leq \mathbb{E}\left[\exp(\beta Y_{\tau_n}) - \exp(\beta Y_t) + \beta \int_t^{\tau_n} \exp(\beta Y_s)(\psi^2 + \frac{1}{2}\varphi^2) \,\mathrm{d}s \middle| \mathcal{F}_t\right],\end{aligned}$$

which we can rewrite as

$$\begin{aligned}&\mathbb{E}\Bigg[\int_t^{\tau_n} \exp(\beta Y_s)|Z_s|^2 \,\mathrm{d}s \middle| \mathcal{F}_t\Bigg] \\ &\leq \mathbb{E}\left[\frac{\exp(\beta Y_{\tau_n}) - \exp(\beta Y_t)}{\beta Y_T - \beta Y_t}(Y_{\tau_n} - Y_t) + \int_t^{\tau_n} \exp(\beta Y_s)(\psi^2 + \frac{1}{2}\varphi^2) \,\mathrm{d}s \middle| \mathcal{F}_t\right] \\ &\leq \left\|\frac{\exp(\beta Y_{\tau_n}) - \exp(\beta Y_t)}{\beta Y_{\tau_n} - \beta Y_t}\right\|_\infty \cdot \|Y_{\tau_n} - Y_t\|_\infty + \exp\left(\beta\|Y\|_\infty\right)\left(\|\psi\|_{BMO(\mathbb{P})}^2 + \frac{1}{2}\|\varphi\|_{BMO(\mathbb{P})}^2\right).\end{aligned}$$

Finally, note that the exponential function is Lipschitz continuous on any interval $[a,b]$ with $\exp(a \vee b)$ as Lipschitz constant, so

$$\left\|\frac{\exp(\beta Y_{\tau_n}) - \exp(\beta Y_t)}{\beta Y_{\tau_n} - \beta Y_t}\right\|_\infty \cdot \|Y_{\tau_n} - Y_t\|_\infty \leq \exp(\beta\|Y\|_\infty) \cdot 2 \cdot \|Y\|_\infty.$$

We obtain by monotone convergence:

$$\begin{aligned}\mathbb{E}&\left[\int_t^T |Z_s|^2 \,\mathrm{d}s \middle| \mathcal{F}_t\right] = \lim_{n\to\infty} \mathbb{E}\left[\int_t^{\tau_n} |Z_s|^2 \,\mathrm{d}s \middle| \mathcal{F}_t\right] \\ &\leq \lim_{n\to\infty} \exp(\beta\|Y\|_\infty)\mathbb{E}\left[\int_t^{\tau_n} \exp(\beta Y_s)|Z_s|^2 \,\mathrm{d}s \middle| \mathcal{F}_t\right] \\ &\leq 2\exp(2\beta\|Y\|_\infty)\|Y\|_\infty + \exp\left(2\beta\|Y\|_\infty\right)\left(\|\psi\|_{BMO(\mathbb{P})}^2 + \frac{1}{2}\|\varphi\|_{BMO(\mathbb{P})}^2\right) \\ &= 2\exp(2(2C+3)\|Y\|_\infty)\|Y\|_\infty + \exp\left(2(2C+3)\|Y\|_\infty\right)\left(\|\psi\|_{BMO(\mathbb{P})}^2 + \frac{1}{2}\|\varphi\|_{BMO(\mathbb{P})}^2\right),\end{aligned}$$

which is finite and increasing in $\|Y\|_\infty$, C, $\|\varphi\|_{BMO(\mathbb{P})}$, $\|\psi\|_{BMO(\mathbb{P})}$. □

A.2 Properties of weak derivatives

In the next result we prove a property of weak derivatives, which should be clear for classical derivatives:

Lemma A.2.1. *Let $X : \mathcal{M} \times \Lambda \to \mathbb{R}$ be a weakly differentiable mapping, where $(\mathcal{M}, \mathcal{A}, \rho)$ is some measure space with finite measure ρ and $\Lambda \subseteq \mathbb{R}^N$ is open, $N \in \mathbb{N}$.*
Let $h \in (0, \infty)$, $v \in \mathbb{R}^N$ and Λ^h the open set of all $\lambda \in \Lambda$ such that $\overline{B_h(\lambda)} \subseteq \Lambda$. Then

$$\int_0^h \frac{\mathrm{d}}{\mathrm{d}\lambda} X(\omega, \lambda_0 + tv)v \,\mathrm{d}t = X(\omega, \lambda_0 + hv) - X(\omega, \lambda_0)$$

for almost all $\lambda_0 \in \Lambda^{h|v|}$, for almost all $\omega \in \mathcal{M}$.

Proof. Choose an $\omega \in \Omega$ s.t. $X(\omega, \cdot)$ is weakly differentiable with a weak derivative $\frac{\mathrm{d}}{\mathrm{d}\lambda} X(\omega, \cdot)$. Define a map $F : \Lambda^{h|v|} \to \mathbb{R}$ via

$$F(\lambda_0) := \int_0^h \frac{\mathrm{d}}{\mathrm{d}\lambda} X(\omega, \lambda_0 + tv)v \,\mathrm{d}t - (X(\omega, \lambda_0 + hv) - X(\omega, \lambda_0)) .$$

Note that this definition makes sense since $(t, \lambda_0) \mapsto \frac{\mathrm{d}}{\mathrm{d}\lambda} X(\omega, \lambda_0 + tv)$ is locally integrable due to

$$\int_0^h \int_{\Lambda^{h|v|}\cap K} \left| \frac{\mathrm{d}}{\mathrm{d}\lambda} X(\omega, \lambda_0 + tv) \right| \mathrm{d}\lambda_0 \,\mathrm{d}t \le \int_0^h \int_{\Lambda\cap\overline{B_{h|v|}(K)}} \left| \frac{\mathrm{d}}{\mathrm{d}\lambda} X(\omega, \lambda_0) \right| \mathrm{d}\lambda_0 \,\mathrm{d}t =$$

$$= h \int_{\Lambda\cap\overline{B_{h|v|}(K)}} \left| \frac{\mathrm{d}}{\mathrm{d}\lambda} X(\omega, \lambda_0) \right| \mathrm{d}\lambda_0 < \infty,$$

for compacts $K \subseteq \mathbb{R}^N$, such that the closed $h|v|$ - neighborhood $\overline{B_{h|v|}(K)}$ of K is also compact obviously.
We want to show that F is a.e. equal 0. For this purpose it is sufficient to show $\int_{\Lambda^{h|v|}} F(\lambda_0)\varphi(\lambda_0) \,\mathrm{d}\lambda_0 = 0$ for every test function $\varphi \in C_c^\infty(\Lambda^{h|v|})$. Now, let φ be such a function. Then using Fubini's theorem

$$\int_{\Lambda^{h|v|}} \int_0^h \varphi(\lambda_0) \frac{\mathrm{d}}{\mathrm{d}\lambda} X(\omega, \lambda_0 + tv)v \,\mathrm{d}t \,\mathrm{d}\lambda_0 = \int_0^h \int_{\Lambda^{h|v|}} \varphi(\lambda_0) \frac{\mathrm{d}}{\mathrm{d}\lambda} X(\omega, \lambda_0 + tv) \,\mathrm{d}\lambda_0 v \,\mathrm{d}t.$$

Using simple transformations and the definition of weak derivatives this is equal to

$$\int_0^h \int_{\Lambda^{h|v|}+tv} \varphi(\lambda_0 - tv) \frac{\mathrm{d}}{\mathrm{d}\lambda} X(\omega, \lambda_0) \,\mathrm{d}\lambda_0 v \,\mathrm{d}t = \int_0^h \int_\Lambda \varphi(\lambda_0 - tv) \frac{\mathrm{d}}{\mathrm{d}\lambda} X(\omega, \lambda_0) \,\mathrm{d}\lambda_0 v \,\mathrm{d}t =$$

$$= - \int_0^h \int_\Lambda X(\omega, \lambda_0) \frac{\mathrm{d}}{\mathrm{d}\lambda} \varphi(\lambda_0 - tv) \,\mathrm{d}\lambda_0 v \,\mathrm{d}t = - \int_\Lambda \int_0^h X(\omega, \lambda_0) \frac{\mathrm{d}}{\mathrm{d}\lambda} \varphi(\lambda_0 - tv)v \,\mathrm{d}t \,\mathrm{d}\lambda_0,$$

where we used Fubini's theorem in the last step. Using simple transformations this value can be rewritten as

$$\int_\Lambda X(\omega, \lambda_0) \int_0^h \frac{\mathrm{d}}{\mathrm{d}\lambda} \varphi(\lambda_0 - tv)(-v) \,\mathrm{d}t \,\mathrm{d}\lambda_0 = \int_\Lambda X(\omega, \lambda_0) \left(\varphi(\lambda_0 - hv) - \varphi(\lambda_0)\right) \mathrm{d}\lambda_0 =$$

$$= \int_\Lambda X(\omega, \lambda_0)\varphi(\lambda_0 - hv) \,\mathrm{d}\lambda_0 - \int_\Lambda X(\omega, \lambda_0)\varphi(\lambda_0) \,\mathrm{d}\lambda_0 =$$

$$= \int_{\Lambda^{h|v|}+hv} X(\omega, \lambda_0)\varphi(\lambda_0 - hv) \,\mathrm{d}\lambda_0 - \int_{\Lambda^{h|v|}} X(\omega, \lambda_0)\varphi(\lambda_0) \,\mathrm{d}\lambda_0 =$$

$$= \int_{\Lambda^{h|v|}} X(\omega, \lambda_0 + hv)\varphi(\lambda_0)\,\mathrm{d}\lambda_0 - \int_{\Lambda^{h|v|}} X(\omega, \lambda_0)\varphi(\lambda_0)\,\mathrm{d}\lambda_0 =$$

$$= \int_{\Lambda^{h|v|}} \left(X(\omega, \lambda_0 + hv) - X(\omega, \lambda_0)\right)\varphi(\lambda_0)\,\mathrm{d}\lambda_0.$$

This already implies $\int_{\Lambda^{h|v|}} F(\lambda_0)\varphi(\lambda_0)\,\mathrm{d}\lambda_0 = 0$. □

For the following result define for measurable $\xi : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$

$$M_{\xi,x} := \inf \{L \geq 0 \,|\, \forall v \in \mathbb{R}^n : \, |\xi(\omega, x+v) - \xi(\omega, x)| \leq L|v| \text{ for a.a. } x \in \mathbb{R}^n, \omega \in \Omega\}.$$

Unlike the constant

$$L_{\xi,x} := \inf \left\{L \geq 0 \,|\, \text{for a.a. } \omega \in \Omega : \, |\xi(\omega, x) - \xi(\omega, x')| \leq L|x - x'| \text{ for all } x, x' \in \mathbb{R}^n\right\}$$

this new constant $M_{\xi,x}$ can be finite even for ξ which are not Lipschitz continuous. Also, $M_{\xi,x} = M_{\tilde{\xi},x}$ if $\tilde{\xi} = \xi$ a.e., which is a property $L_{\xi,x}$ does not have. Instead, it satisfies the weaker property $L_{\xi,x} = L_{\tilde{\xi},x}$ if $\tilde{\xi}(\omega, \cdot) = \xi(\omega, \cdot)$ on the whole $\mathbb{R}^n$ for a.a. $\omega \in \Omega$.

Lemma A.2.2. *Let $\xi : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ be measurable such that $M_{\xi,x} < \infty$. Then there exists a measurable $\tilde{\xi} : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ with $\xi = \tilde{\xi}$ a.e. and*

$$\left|\tilde{\xi}(\omega, x') - \tilde{\xi}(\omega, x)\right| \leq M_{\xi,x}|x' - x| \qquad \forall \omega \in \Omega, x', x \in \mathbb{R}^n.$$

Proof. Due to $M_{\xi,x} < \infty$ we have $|\xi(\omega, x+v) - \xi(\omega, x)| \leq M_{\xi,x}|v|$ for almost all $x, v \in \mathbb{R}^n, \omega \in \Omega$. Therefore, $|\xi(\omega, y) - \xi(\omega, x)| \leq M_{\xi,x}|y - x|$ for almost all $x, y \in \mathbb{R}^n, \omega \in \Omega$ and, so, for almost all $x \in \mathbb{R}^n, \omega \in \Omega$ the mapping $y \longmapsto \xi(\omega, y) - \xi(\omega, x)$ ist locally integrable. Thus $\xi(\omega, \cdot)$ is locally integrable for almost all $\omega \in \Omega$ and we can consider the expression

$$\frac{1}{|B_\varepsilon(0)|} \int_{B_\varepsilon(x)} \xi(\omega, y)\,\mathrm{d}y =: \xi_\varepsilon(\omega, x),$$

for any $\varepsilon > 0$, which according to Lebesgue's differentiation theorem converges for almost all ω, x to $\xi(\omega, x)$ as $\varepsilon \to 0$.

We claim that for every $\varepsilon > 0$ this $\xi_\varepsilon(\omega, \cdot)$ is truly Lipschitz continuous for almost all $\omega \in \Omega$: For almost all $\omega \in \Omega$, but all $x, x' \in \mathbb{R}^n$:

$$|\xi_\varepsilon(\omega, x) - \xi_\varepsilon(\omega, x')| = \left|\frac{1}{|B_\varepsilon(0)|} \int_{B_\varepsilon(0)} \xi(\omega, x+z)\,\mathrm{d}z - \frac{1}{|B_\varepsilon(0)|} \int_{B_\varepsilon(0)} \xi(\omega, x'+z)\,\mathrm{d}z\right| \leq$$

$$\leq \frac{1}{|B_\varepsilon(0)|} \int_{B_\varepsilon(0)} \left|\xi(\omega, x+z) - \xi(\omega, x'+z)\right|\,\mathrm{d}z \leq \frac{1}{|B_\varepsilon(0)|} \int_{B_\varepsilon(0)} M_{\xi,x}|x - x'|\,\mathrm{d}z = M_{\xi,x}|x - x'|,$$

due to the fact that the shift $x + z - (x' + z) = x - x'$ does not depend on the running variable z and due to the definition of $M_{\xi,x}$. This argument explains, by the way, why $M_{\xi,x}$ was defined the way it was in the first place. ✓

We claim that $\hat{\xi}(\omega, \cdot) := \limsup_{n\to\infty} \xi_{\frac{1}{n}}(\omega, \cdot)$, where the lim sup is applied component-wise, must also be truly Lipschitz continuous for almost all ω: For every component $i = 1, \ldots, m$ and all $x, x' \in \mathbb{R}^n$

$$\xi^{(i)}_{\frac{1}{n}}(\omega, x) - \xi^{(i)}_{\frac{1}{n}}(\omega, x') \leq \left|\xi_{\frac{1}{n}}(\omega, x) - \xi_{\frac{1}{n}}(\omega, x')\right| \leq M_{\xi,x}|x - x'|$$

implies

$$\limsup_{n\to\infty} \xi^{(i)}_{\frac{1}{n}}(\omega, x) \leq M_{\xi,x}|x - x'| + \limsup_{n\to\infty} \xi^{(i)}_{\frac{1}{n}}(\omega, x')$$

which by exchanging the roles of x and x' leads to

$$\left|\limsup_{n\to\infty} \xi^{(i)}_{\frac{1}{n}}(\omega, x) - \limsup_{n\to\infty} \xi^{(i)}_{\frac{1}{n}}(\omega, x')\right| \leq M_{\xi,x}|x - x'| \qquad \forall x, x' \in \mathbb{R}^n$$

for a.a. $\omega \in \Omega$. So, $\hat{\xi}(\omega, \cdot)$ is Lipschitz continuous for a.a. ω, but we do not have the right Lipschitz constant yet.

Clearly, $\hat{\xi}$ coincides with ξ up to a null set (Lebesgue's differentiation theorem). Therefore, the property $|\hat{\xi}(\omega, y) - \hat{\xi}(\omega, x)| \leq M_{\xi,x}|y - x|$ for almost all $x, y \in \mathbb{R}^n, \omega \in \Omega$ and continuity of $\hat{\xi}(\omega, \cdot)$ implies

$$|\hat{\xi}(\omega, y) - \hat{\xi}(\omega, x)| \leq M_{\xi,x}|y - x| \qquad \forall x, y \in \mathbb{R}^n$$

for a.a. $\omega \in \Omega$. So, by setting $\hat{\xi}(\omega, \cdot)$ to zero for all those $\omega \in \Omega$ for which this property does not hold we obtain the desired $\tilde{\xi}$. □

Proof of Lemma 2.1.4. "$\Longrightarrow$": Assume without loss of generality $L_{\xi,x} < \infty$. According to Lemma A.3.1 (applied ω - wise) ξ is weakly differentiable w.r.t. $x \in \mathbb{R}^n$. Using Lemma A.2.1 we have for arbitrarily chosen $v \in S^{n-1}$ and $h > 0$:

$$\frac{1}{h}\int_0^h \frac{\mathrm{d}}{\mathrm{d}x}\xi(\omega, x_0 + tv)v \,\mathrm{d}t = \frac{1}{h}\left(\xi(\omega, x_0 + hv) - \xi(\omega, x_0)\right)$$

for almost all $x_0 \in \mathbb{R}^n$, for almost all $\omega \in \Omega$. Using Lipschitz continuity of $\xi(\omega, \cdot)$ we obtain

$$\left|\frac{1}{h}\int_0^h \frac{\mathrm{d}}{\mathrm{d}x}\xi(\omega, x_0 + tv)v \,\mathrm{d}t\right| \leq L_{\xi,x}$$

for almost all $x_0 \in \mathbb{R}^n$, for almost all $\omega \in \Omega$. Letting $h \to 0$ and using the fundamental Theorem of Lebesgue integral calculus we obtain

$$\left|\frac{\mathrm{d}}{\mathrm{d}x}\xi(\omega, x_0)\right|_v = \left|\frac{\mathrm{d}}{\mathrm{d}x}\xi(\omega, x_0)v\right| \leq L_{\xi,x}$$

for almost all $x_0 \in \mathbb{R}^n$, for almost all $\omega \in \Omega$. This implies

$$\sup_{v \in S^{n-1}} \operatorname{ess\,sup}_{\omega\in\Omega, x\in\mathbb{R}^n} \left|\frac{\mathrm{d}}{\mathrm{d}x}\xi(\omega, x)\right|_v \leq L_{\xi,x}.$$

Buy plugging in the n different canonical unit vectors for v we also see from this inequality that $\frac{\mathrm{d}}{\mathrm{d}x}\xi$ is bounded up to a null set.

"$\Longleftarrow$": Assume ξ is weakly differentiable w.r.t. x such that $\frac{\mathrm{d}}{\mathrm{d}x}\xi$ is bounded up to a null set and, thereby,

$$C := \sup_{v \in S^{n-1}} \operatorname{ess\,sup}_{\omega\in\Omega, x\in\mathbb{R}^n} \left|\frac{\mathrm{d}}{\mathrm{d}x}\xi(\omega, x)\right|_v < \infty$$

holds. For $v \in S^{n-1}$ let $F_v \subseteq \Omega \times \mathbb{R}^n$ be a set of full measure (i.e. its complement is a null set) such that $\left|\frac{\mathrm{d}}{\mathrm{d}x}\xi(\omega, x)\right|_v \leq C$ for all $(\omega, x) \in F_v$. Let S' be a countable but dense subset of S^{n-1} and define $F := \bigcap_{v\in S'} F_v$. Then F is also a set of full measure and

$$\sup_{v \in S^{n-1}} \left|\frac{\mathrm{d}}{\mathrm{d}x}\xi(\omega, x)\right|_v = \sup_{v \in S'} \left|\frac{\mathrm{d}}{\mathrm{d}x}\xi(\omega, x)\right|_v \leq C \qquad \forall (\omega, x) \in F.$$

Therefore, by modifying $\frac{\mathrm{d}}{\mathrm{d}x}\xi$ by setting it to 0 on a null set we can actually assume without loss of generality that $\sup_{v\in S^{n-1}} \left|\frac{\mathrm{d}}{\mathrm{d}x}\xi(\omega,x)\right|_v \leq C$ for all ω, x.
Let $h > 0$ and $v \in S^{n-1}$ be arbitrary. Applying Lemma A.2.1:

$$|\xi(\omega, x_0 + hv) - \xi(\omega, x_0)| = \left|\int_0^h \frac{\mathrm{d}}{\mathrm{d}x}\xi(\omega, x_0 + tv)v\,\mathrm{d}t\right| \leq \int_0^h \left|\frac{\mathrm{d}}{\mathrm{d}x}\xi(\omega, x_0 + tv)v\right| \mathrm{d}t \leq Ch = C|hv|,$$

for almost all $x_0 \in \mathbb{R}^n$, $\omega \in \Omega$. In other words for every $z \in \mathbb{R}^n$:

$$|\xi(\omega, x_0 + z) - \xi(\omega, x_0)| \leq C|z|, \tag{A.3}$$

for almost all $x_0 \in \mathbb{R}^n$, $\omega \in \Omega$. Therefore, we can apply Lemma A.2.2 to obtain a $\tilde{\xi} = \xi$ a.e. with $L_{\tilde{\xi},x} \leq C$. □

Lemma A.2.3. *For $n \in \mathbb{N}$ let $X : \mathcal{M} \times \mathbb{R}^n \to \mathbb{R}$ be some measurable and weakly differentiable mapping, where $(\mathcal{M}, \mathcal{A}, \rho)$ is a measure space with finite measure ρ.*
Let $p \geq 1$. If $\int_{\mathcal{M}} |X(\cdot,\lambda)|^p \,\mathrm{d}\rho < \infty$ for all $\lambda \in \mathbb{R}^n$ and

$$\sup_{v\in S^{n-1}} \operatorname{ess\,sup}_{\lambda\in\mathbb{R}^n} \int_{\mathcal{M}} \left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X(\cdot,\lambda)\right|_v^p \mathrm{d}\rho < \infty,$$

then $\lambda \mapsto \int_{\mathcal{M}} |X(\cdot,\lambda)|^p \,\mathrm{d}\rho$ is locally integrable.

Proof. Note that $\operatorname{ess\,sup}_{\lambda\in\mathbb{R}^n} \int_{\mathcal{M}} \left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X(\cdot,\lambda)\right|_v^p \mathrm{d}\rho$ is finite for every canonical unit vector v and so the value $\operatorname{ess\,sup}_{\lambda\in\mathbb{R}^n} \int_{\mathcal{M}} \left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}X(\cdot,\lambda)\right\|^p \mathrm{d}\rho$ is finite for some and, therefore, for every norm $\|\cdot\|$ on $\mathbb{R}^{1\times n}$, in particular the operator norm.
According to Lemma A.2.1 we have

$$\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\lambda}X(\omega, \lambda + tv)v\,\mathrm{d}t = X(\omega, \lambda + v) - X(\omega, \lambda)$$

for almost all $\omega \in \Omega, \lambda, v \in \mathbb{R}^n$. Let λ_0 be such that the above property holds for this fixed $\lambda = \lambda_0$ and almost all ω, v. Since $\int_{\mathcal{M}} |X(\cdot,\lambda_0)|^p \,\mathrm{d}\rho < \infty$ it is sufficient to show local integrability of the mapping

$$v \longmapsto \int_{\mathcal{M}} \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\lambda}X(\cdot, \lambda_0 + tv)v\,\mathrm{d}t\right|^p \mathrm{d}\rho =_{\text{a.e.}} \int_{\mathcal{M}} |X(\cdot,\lambda_0 + v) - X(\cdot,\lambda_0)|^p \,\mathrm{d}\rho.$$

Let $K \subset \mathbb{R}^n$ be a compact set. Using Jensen's inequality and Fubini's theorem together with simple estimates

$$\int_K \int_{\mathcal{M}} \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\lambda}X(\cdot, \lambda_0 + tv)v\,\mathrm{d}t\right|^p \mathrm{d}\rho\,\mathrm{d}v \leq \int_K \int_{\mathcal{M}} \int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X(\cdot, \lambda_0 + tv)v\right|^p \mathrm{d}t\,\mathrm{d}\rho\,\mathrm{d}v =$$

$$= \int_K \int_0^1 \int_{\mathcal{M}} \left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X(\cdot, \lambda_0 + tv)v\right|^p \mathrm{d}\rho\,\mathrm{d}t\,\mathrm{d}v \leq \int_K \int_0^1 \int_{\mathcal{M}} \sup_{w\in S^{n-1}} \left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X(\cdot, \lambda_0 + tv)\right|_w^p |v|^p \,\mathrm{d}\rho\,\mathrm{d}t\,\mathrm{d}v \leq$$

$$\leq \int_K \int_0^1 \int_{\mathcal{M}} \sup_{w\in S^{n-1}} \left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X(\cdot, \lambda_0 + tv)\right|_w^p \mathrm{d}\rho\,\mathrm{d}t\,\mathrm{d}v \left(\sup_{v\in K} |v|^p\right).$$

Using Fubini's theorem again this value is seen to be equal to

$$\int_0^1 \int_K \int_{\mathcal{M}} \sup_{w\in S^{n-1}} \left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X(\cdot, \lambda_0 + tv)\right|_w^p \mathrm{d}\rho\,\mathrm{d}v\,\mathrm{d}t \left(\sup_{v\in K} |v|^p\right) \leq$$

$$\leq \int_0^1 \int_K \left(\operatorname{ess\,sup}_{\lambda \in \mathbb{R}^n} \int_{\mathcal{M}} \sup_{w \in S^{n-1}} \left| \frac{\mathrm{d}}{\mathrm{d}\lambda} X(\cdot, \lambda) \right|_w^p \mathrm{d}\rho \right) \mathrm{d}v \, \mathrm{d}t \left(\sup_{v \in K} |v|^p \right) \leq$$

$$\leq \left(\operatorname{ess\,sup}_{\lambda \in \mathbb{R}^n} \int_{\mathcal{M}} \sup_{w \in S^{n-1}} \left| \frac{\mathrm{d}}{\mathrm{d}\lambda} X(\cdot, \lambda) \right|_w^p \mathrm{d}\rho \right) \cdot |K| \cdot \left(\sup_{v \in K} |v|^p \right) < \infty.$$

$\square$

Lemma A.2.4. *For an arbitrary probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let $X : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be measurable, s.t.*

- *X is weakly differentiable w.r.t. λ,*
- *$\mathbb{E}\left[|X(\cdot, \lambda)|\right] < \infty$ for all $\lambda \in \mathbb{R}^n$ and*
- *$\operatorname{ess\,sup}_{\lambda \in \mathbb{R}^n} \sup_{v \in S^{n-1}} \mathbb{E}\left[\left| \frac{\mathrm{d}}{\mathrm{d}\lambda} X(\cdot, \lambda) \right|_v \right] < \infty$.*

Let also $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra. Then the mapping $(\omega, \lambda) \mapsto \mathbb{E}[X(\cdot, \lambda)|\mathcal{G}](\omega)$ is measurable and weakly differentiable w.r.t. λ and $\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbb{E}[X(\cdot, \lambda)|\mathcal{G}] = \mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}\lambda} X(\cdot, \lambda)|\mathcal{G}\right]$.

Proof of Lemma A.2.4. Define mappings $Y : \Omega \times \mathbb{R}^n \to \mathbb{R}$ and $Z : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ via $Y(\omega, \lambda) := \mathbb{E}\left[X(\cdot, \lambda)|\mathcal{G}\right](\omega)$ and $Z(\omega, \lambda) := \mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}\lambda} X(\cdot, \lambda)|\mathcal{G}\right](\omega)$. Note that Y is measurable, since $Y \mathbf{1}_{\Omega \times B_R(\lambda_0)}$ can be seen as a conditional expectation w.r.t. $\mathcal{G} \otimes \mathcal{B}(B_R(\lambda_0))$ on the probability space

$$\left(\Omega \times B_R(\lambda_0), \, \mathcal{F} \otimes \mathcal{B}(B_R(\lambda_0)), \, \mathbb{P} \otimes \frac{1}{|B_R(\lambda_0)|} \rho_{B_R(\lambda_0)} \right),$$

where $\lambda_0 \in \mathbb{R}^n$, $R > 0$ are arbitrary and $\rho_{B_R(\lambda_0)}$ is the Lebesgue measure on the ball $B_R(\lambda_0)$.
We now claim that Z is the weak derivative of Y: Take a real valued test function $\varphi \in C_c^\infty(\mathbb{R}^n)$. We have

$$\int_{\mathbb{R}^n} \varphi(\lambda) Z(\cdot, \lambda) \, \mathrm{d}\lambda = \int_{\mathbb{R}^n} \varphi(\lambda) \mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}\lambda} X(\cdot, \lambda) \middle| \mathcal{G} \right] \mathrm{d}\lambda = \mathbb{E}\left[\int_{\mathbb{R}^n} \varphi(\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda} X(\cdot, \lambda) \, \mathrm{d}\lambda \middle| \mathcal{G} \right] =$$

$$= \mathbb{E}\left[- \int_{\mathbb{R}^n} X(\cdot, \lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda} \varphi(\lambda) \, \mathrm{d}\lambda \middle| \mathcal{G} \right] = - \int_{\mathbb{R}^n} \mathbb{E}\left[X(\cdot, \lambda) \middle| \mathcal{G} \right] \frac{\mathrm{d}}{\mathrm{d}\lambda} \varphi(\lambda) \, \mathrm{d}\lambda = - \int_{\mathbb{R}^n} Y(\cdot, \lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda} \varphi(\lambda) \, \mathrm{d}\lambda,$$

where we used Fubini's theorem. All integrals make sense according to Lemma A.2.3. $\square$

Let from now on $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]})$ be as in subsection 2.1.1.

Lemma A.2.5. *Let $Z : [0, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}$ be measurable, s.t.*

- *Z is weakly differentiable w.r.t. $\lambda \in \mathbb{R}^n$,*
- *$\mathbb{E}\left[\int_0^T |Z_s(\cdot, \lambda)| \, \mathrm{d}s \right] < \infty$ for all $\lambda \in \mathbb{R}^n$ and*
- *$\operatorname{ess\,sup}_{\lambda \in \mathbb{R}^n} \sup_{v \in S^{n-1}} \mathbb{E}\left[\int_0^T \left| \frac{\mathrm{d}}{\mathrm{d}\lambda} Z_s(\cdot, \lambda) \right|_v \mathrm{d}s \right] < \infty$.*

Then the mapping $X := \int_0^T Z_s \, \mathrm{d}s : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is measurable, weakly differentiable w.r.t. $\lambda \in \mathbb{R}^n$ and $\frac{\mathrm{d}}{\mathrm{d}\lambda} X(\cdot, \lambda) = \int_0^T \frac{\mathrm{d}}{\mathrm{d}\lambda} Z_s(\cdot, \lambda) \, \mathrm{d}s$.

Proof. Define a new probability space $\left([0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{F}, \frac{1}{T} \rho_{[0,T]} \otimes \mathbb{P}\right)$, where $\rho_{[0,T]}$ is the Lebesgue measure on $[0, T]$ and $\mathcal{B}([0, T])$ is the σ-Algebra of Borel-measurable subsets of $[0, T]$. Define further $\mathcal{G} := \{\emptyset, [0, T]\} \otimes \mathcal{F}$ and apply Lemma A.2.4. $\square$

Lemma A.2.6. *Let $Z : [0, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^{1 \times d}$ be progressively measurable, s.t.*

- *Z is weakly differentiable w.r.t. $\lambda \in \mathbb{R}^n$,*
- $\mathbb{E}\left[\int_0^T |Z_s(\cdot,\lambda)|^2 \,\mathrm{d}s\right] < \infty$ *for all $\lambda \in \mathbb{R}^n$ and*
- $\operatorname{ess\,sup}_{\lambda\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \mathbb{E}\left[\int_0^T \left|\frac{\mathrm{d}}{\mathrm{d}\lambda} Z_s(\cdot,\lambda)\right|_v^2 \,\mathrm{d}s\right] < \infty.$

Then the mapping $X := \int_0^T Z_s \,\mathrm{d}W_s : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is measurable, weakly differentiable w.r.t. $\lambda \in \mathbb{R}^n$ and $\frac{\mathrm{d}}{\mathrm{d}\lambda} X(\cdot,\lambda)v = \int_0^T \frac{\mathrm{d}}{\mathrm{d}\lambda} Z_s(\cdot,\lambda)v \,\mathrm{d}W_s$ for all $v \in \mathbb{R}^n$.

Proof of Lemma A.2.6. Measurability of X follows from the fact that the stochastic integral $\int_0^T Z_s \,\mathrm{d}W_s$ can be defined as an a.e. limit of integrals over simple progressive processes Z^n and such integrals are measurable, since $Z_s^n(\cdot,\cdot)$ must be measurable for every $s \in [0,T]$.
It remains to verify that $\int_0^T \frac{\mathrm{d}}{\mathrm{d}\lambda} Z_s(\cdot,\lambda)v \,\mathrm{d}W_s$ is a weak derivative of X in direction v. To see this, take a test function $\varphi \in C_c^\infty(\mathbb{R}^n)$ and choose any $v \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} \varphi(\lambda) \int_0^T \frac{\mathrm{d}}{\mathrm{d}\lambda} Z_s(\cdot,\lambda)v \,\mathrm{d}W_s \,\mathrm{d}\lambda = \int_{\mathbb{R}^n} \int_0^T \varphi(\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda} Z_s(\cdot,\lambda)v \,\mathrm{d}W_s \,\mathrm{d}\lambda =$$

$$= \int_0^T \int_{\mathbb{R}^n} \varphi(\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda} Z_s(\cdot,\lambda)v \,\mathrm{d}\lambda \,\mathrm{d}W_s = -\int_0^T \int_{\mathbb{R}^n} Z_s(\cdot,\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda}\varphi(\lambda)v \,\mathrm{d}\lambda \,\mathrm{d}W_s =$$

$$= -\int_{\mathbb{R}^n} \int_0^T Z_s(\cdot,\lambda) \,\mathrm{d}W_s \frac{\mathrm{d}}{\mathrm{d}\lambda}\varphi(\lambda)v \,\mathrm{d}\lambda = -\int_{\mathbb{R}^n} X(\cdot,\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda}\varphi(\lambda)v \,\mathrm{d}\lambda,$$

where we used continuity and linearity of the stochastic integral twice. All integrals make sense according to Lemma A.2.3 and using Itô isometry: For instance

$$\mathbb{E}\left[|X(\cdot,\lambda)|^2\right] = \mathbb{E}\left[\left|\int_0^T Z_s(\cdot,\lambda) \,\mathrm{d}W_s\right|^2\right] = \mathbb{E}\left[\int_0^T |Z_s(\cdot,\lambda)|^2 \,\mathrm{d}s\right]$$

is locally integrable w.r.t. λ according to Lemma A.2.3. So,

$$\lambda \mapsto \mathbb{E}\left[|X(\cdot,\lambda)|\right] = \mathbb{E}\left[\left|\int_0^T Z_s(\cdot,\lambda) \,\mathrm{d}W_s\right|\right] \leq \left(\mathbb{E}\left[\left|\int_0^T Z_s(\cdot,\lambda) \,\mathrm{d}W_s\right|^2\right]\right)^{\frac{1}{2}}$$

must be locally integrable as well. $\square$

Conversely, we can also show Lemma A.2.7. For this result we denote by $\mathbb{E}_0[X]$ the conditional expectation $\mathbb{E}[X|\mathcal{F}_0]$, which is well-defined for all integrable random variables X.

Lemma A.2.7. *Let $X : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be measurable, s.t.*

- *X is weakly differentiable w.r.t. λ,*
- $\mathbb{E}\left[|X(\cdot,\lambda)|^2\right] < \infty$ *for all $\lambda \in \mathbb{R}^n$ and*
- $\operatorname{ess\,sup}_{\lambda\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \mathbb{E}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda} X(\cdot,\lambda)\right|_v^2\right] < \infty.$

Then the unique progressively measurable process $Z : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^{1\times d}$ such that $X = \mathbb{E}_0[X] + \int_0^T Z_s \,\mathrm{d}W_s$ has the property that it is weakly differentiable w.r.t. λ s.t. $\frac{\mathrm{d}}{\mathrm{d}\lambda} X(\cdot,\lambda)v = \mathbb{E}_0\left[\frac{\mathrm{d}}{\mathrm{d}\lambda} X(\cdot,\lambda)v\right] + \int_0^T \frac{\mathrm{d}}{\mathrm{d}\lambda} Z_s(\cdot,\lambda)v \,\mathrm{d}W_s$ for all $v \in \mathbb{R}^n$.

Proof of Lemma A.2.7. Existence of Z follows from the Itô representation formula, which is applied to X. It can also be applied to $\frac{\mathrm{d}}{\mathrm{d}\lambda}X$, yielding a second progressively measurable process $\tilde{Z} : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^{d\times n}$. It remains to show that $\tilde{Z}$ is a weak derivative of Z. To verify this, take a test function $\varphi \in C_c^\infty(\mathbb{R}^n)$ and choose any $v \in \mathbb{R}^n$. Then we have using continuity and linearity of the stochastic integral together with Lemma A.2.4

$$\int_0^T \int_{\mathbb{R}^n} \varphi(\lambda)\tilde{Z}_s(\cdot,\lambda)v \,\mathrm{d}\lambda \,\mathrm{d}W_s = \int_{\mathbb{R}^n} \varphi(\lambda) \int_0^T \tilde{Z}_s(\cdot,\lambda)v \,\mathrm{d}W_s \,\mathrm{d}\lambda =$$

$$= \int_{\mathbb{R}^n} \varphi(\lambda)\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}X(\cdot,\lambda) - \mathbb{E}_0\left[\frac{\mathrm{d}}{\mathrm{d}\lambda}X(\cdot,\lambda)\right]\right) v \,\mathrm{d}\lambda = -\int_{\mathbb{R}^n} \left(X(\cdot,\lambda) - \mathbb{E}_0\left[X(\cdot,\lambda)\right]\right) \frac{\mathrm{d}}{\mathrm{d}\lambda}\varphi(\lambda)v \,\mathrm{d}\lambda =$$

$$= -\int_{\mathbb{R}^n} \int_0^T Z_s(\cdot,\lambda) \,\mathrm{d}W_s \frac{\mathrm{d}}{\mathrm{d}\lambda}\varphi(\lambda)v \,\mathrm{d}\lambda = \int_0^T \left(-\int_{\mathbb{R}^n} Z_s(\cdot,\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}\varphi(\lambda)v \,\mathrm{d}\lambda\right) \mathrm{d}W_s,$$

which already implies $\int_{\mathbb{R}^n} \varphi(\lambda)\tilde{Z}_s(\cdot,\lambda)v \,\mathrm{d}\lambda = -\int_{\mathbb{R}^n} Z_s(\cdot,\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}\varphi(\lambda)v \,\mathrm{d}\lambda$. All integrals make sense according to Lemma A.2.3 (see proof of Lemma A.2.6 for details). □

Lemma A.2.8. *Let $X : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be measurable and $V : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}$ progressively measurable s.t.*

- *X and V are weakly differentiable w.r.t. $\lambda \in \mathbb{R}^n$,*
- *$\mathbb{E}\left[|X(\cdot,\lambda)|^2\right] < \infty$ for all $\lambda \in \mathbb{R}^n$,*
- *$\operatorname{ess\,sup}_{\lambda\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \mathbb{E}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X(\cdot,\lambda)\right|_v^2\right] < \infty$,*
- *$\mathbb{E}\left[\left(\int_0^T |V_s(\cdot,\lambda)| \,\mathrm{d}s\right)^2\right] < \infty$ for all $\lambda \in \mathbb{R}^n$ and*
- *$\operatorname{ess\,sup}_{\lambda\in\mathbb{R}^n} \sup_{v\in S^{n-1}} \mathbb{E}\left[\left(\int_0^T \left|\frac{\mathrm{d}}{\mathrm{d}\lambda}V_s(\cdot,\lambda)\right|_v \,\mathrm{d}s\right)^2\right] < \infty$.*

Then there exist unique progressive processes $Y, Z : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}, \mathbb{R}^{1\times d}$ s.t.

$$Y_t = X - \int_t^T V_s \,\mathrm{d}s - \int_t^T Z_s \,\mathrm{d}W_s \quad \textit{a.s. for all } t \in [0,T].$$

These Y and Z are both weakly differentiable w.r.t. λ and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_t v = \frac{\mathrm{d}}{\mathrm{d}\lambda}Xv - \int_t^T \frac{\mathrm{d}}{\mathrm{d}\lambda}V_s v \,\mathrm{d}s - \int_t^T \frac{\mathrm{d}}{\mathrm{d}\lambda}Z_s v \,\mathrm{d}W_s \quad \textit{a.s. for all } v \in \mathbb{R}^n, t \in [0,T].$$

Proof of Lemma A.2.8. For each $t \in [0,T]$ define

$$Y_t := \mathbb{E}\left[X - \int_t^T V_s \,\mathrm{d}s \middle| \mathcal{F}_t\right] = \mathbb{E}\left[X - \int_0^T V_s \,\mathrm{d}s \middle| \mathcal{F}_t\right] + \int_0^t V_s \,\mathrm{d}s.$$

The mapping $Y_t : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{F}_t \otimes \mathcal{L}(\mathbb{R}^n)$ - measurable and weakly differentiable w.r.t. λ, such that $\frac{\mathrm{d}}{\mathrm{d}\lambda}Y_t = \mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}\lambda}X - \int_t^T \frac{\mathrm{d}}{\mathrm{d}\lambda}V_s \,\mathrm{d}s \middle| \mathcal{F}_t\right]$, according to Lemma A.2.5 and Lemma A.2.4. Thereby we obtain a process Y, which is continuous in time and, therefore, progressively measurable. Now, define

$$M := X - \int_0^T V_s \,\mathrm{d}s - Y_0.$$

$M : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is measurable and weakly differentiable w.r.t. λ. It is also straightforward to check, that $\mathbb{E}\left[|M(\cdot,\lambda)|^2\right] < \infty$ for all λ and $\operatorname{ess\,sup}_{\lambda\in\mathbb{R}^n}\mathbb{E}\left[\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}M(\cdot,\lambda)\right|^2\right] < \infty$. Therefore, we can apply Lemma A.2.7 and write $M = \int_0^T Z_s\,\mathrm{d}W_s$ with a progressively measurable and weakly differentiable Z. Also,

$$\int_0^T Z_s\,\mathrm{d}W_s = X - \int_0^T V_s\,\mathrm{d}s - Y_0.$$

Applying conditional expectations gives us

$$\int_0^t Z_s\,\mathrm{d}W_s = \mathbb{E}\left[X - \int_0^T V_s\,\mathrm{d}s\middle|\mathcal{F}_t\right] - Y_0 = Y_t - \int_0^t V_s\,\mathrm{d}s - Y_0.$$

Subtracting this equation from the preceding one leads to

$$\int_t^T Z_s\,\mathrm{d}W_s = X - \int_t^T V_s\,\mathrm{d}s - Y_t.$$

We can now differentiate w.r.t. λ according to Lemma A.2.6 and Lemma A.2.5 with the result

$$\int_t^T \frac{\mathrm{d}}{\mathrm{d}\lambda}Z_s v\,\mathrm{d}W_s = \frac{\mathrm{d}}{\mathrm{d}\lambda}Xv - \int_t^T \frac{\mathrm{d}}{\mathrm{d}\lambda}V_s v\,\mathrm{d}s - \frac{\mathrm{d}}{\mathrm{d}\lambda}Y_t v$$

for all $v \in \mathbb{R}^n$. □

Proof of Lemma 2.1.5. First we claim that $\lambda \mapsto \sup_{i\in\mathbb{N}}\int_{\mathcal{M}}|X_i(\lambda,\cdot)|^2\,\mathrm{d}\rho$ is bounded up to a null set by some constant on sets of the form $\Lambda \cap B_\varepsilon(\lambda_0)$ for almost all $\lambda_0 \in \Lambda$, where $\varepsilon > 0$ is arbitrary: We have for a.a. $\lambda \in B_\varepsilon(\lambda_0) \cap \Lambda$ using Lemma A.2.1 and Fubini's theorem:

$$\left|\sqrt{\int_{\mathcal{M}}|X_i(\lambda,\cdot)|^2\,\mathrm{d}\rho} - \sqrt{\int_{\mathcal{M}}|X_i(\lambda_0,\cdot)|^2\,\mathrm{d}\rho}\right| \le \sqrt{\int_{\mathcal{M}}|X_i(\lambda,\cdot) - X_i(\lambda_0,\cdot)|^2\,\mathrm{d}\rho} =$$

$$= \sqrt{\int_{\mathcal{M}}\left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\lambda}X_i(\lambda_0 + s(\lambda-\lambda_0),\cdot)(\lambda-\lambda_0)\,\mathrm{d}s\right|^2\mathrm{d}\rho} \le$$

$$\le \sqrt{\int_{\mathcal{M}}\int_0^1\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_i(\lambda_0 + s(\lambda-\lambda_0),\cdot)(\lambda-\lambda_0)\right|^2\mathrm{d}s\,\mathrm{d}\rho} \le$$

$$\le \sqrt{\int_0^1\int_{\mathcal{M}}\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}X_i(\lambda_0 + s(\lambda-\lambda_0),\cdot)\right|^2|\lambda-\lambda_0|^2\,\mathrm{d}\rho\,\mathrm{d}s} \le \sqrt{C}\,|\lambda-\lambda_0| \le \sqrt{C}\varepsilon.$$

Therefore, $(i,\lambda) \mapsto \int_{\mathcal{M}}|X_i(\lambda,\cdot)|^2\,\mathrm{d}\rho$, $\lambda \in B_\varepsilon(\lambda_0)$, $i \in \mathbb{N}$, must be bounded, since

$$\sup_{i\in\mathbb{N}}\int_{\mathcal{M}}|X_i(\lambda_0,\cdot)|^2\,\mathrm{d}\rho < \infty,$$

because of the $\mathcal{L}^2$-convergence of $X_i(\lambda_0,\cdot)$ to some $X(\lambda_0,\cdot)$ (for all $\lambda_0 \in \Lambda$). ✓

The above means that for every compact set $K \subset \Lambda$ the mapping $K \ni \lambda \mapsto \int_{\mathcal{M}}|X_i(\lambda,\cdot)|^2\,\mathrm{d}\rho$ is bounded up to a null set by some constant which does not depend on i. The same must hold for $\lambda \mapsto \int_{\mathcal{M}}|X(\lambda,\cdot)|^2\,\mathrm{d}\rho$ by passing to the limit $i \to \infty$.

Now, define $\tilde{X}(\lambda,\omega) := \limsup_{i\to\infty}X_i(\lambda,\omega)$. Clearly, $\tilde{X}$ is measurable and $\tilde{X}(\lambda,\cdot) = X(\lambda,\cdot)$ a.e. for every $\lambda \in \Lambda$. ✓

The above statements imply that $X_i \to \tilde{X}$ in $\mathcal{L}^2$ on sets $K \times \mathcal{M}$, where $K \subset \Lambda$ is compact.

For $\lambda_0 \in \mathbb{R}^N$ and $\varepsilon > 0$ let $\mathbb{H}_{\varepsilon,\lambda_0,\delta}$ be the Hilbert space of real valued measurable functions Y on $S_{\varepsilon,\lambda_0} := (\Lambda \cap B_\varepsilon(\lambda_0)) \times \mathcal{M}$ s.t. Y is weakly differentiable up to order δ and

$$\sum_{0\le|\alpha|\le\delta} \int_{\Lambda\cap B_\varepsilon(\lambda_0)} \int_{\mathcal{M}} |D^\alpha_\lambda Y(\lambda,\cdot)|^2 \,\mathrm{d}\rho\,\mathrm{d}\lambda < \infty.$$

Obviously (X_i) is a bounded sequence in $\mathbb{H}_{\varepsilon,\lambda_0,\delta}$. We claim that X must be in $\mathbb{H}_{\varepsilon,\lambda_0,\delta}$, too. Let $\alpha \in \mathbb{N}^N$ be a multi-index s.t. $1 \le |\alpha| \le \delta$. We have

$$\int_{\Lambda\cap B_\varepsilon(\lambda_0)} \int_{\mathcal{M}} D^\alpha_\lambda X_i(\lambda,\cdot)\varphi(\lambda,\cdot)\,\mathrm{d}\rho\,\mathrm{d}\lambda = (-1)^{|\alpha|} \int_{\Lambda\cap B_\varepsilon(\lambda_0)} \int_{\mathcal{M}} X_i(\lambda,\cdot) D^\alpha_\lambda \varphi(\lambda,\cdot)\,\mathrm{d}\rho\,\mathrm{d}\lambda$$

for all smooth $\varphi \in \mathbb{H}_{\varepsilon,\lambda_0,\delta}$ s.t. the support of $\varphi(\cdot,\omega)$ is a subset of $B_\varepsilon(\lambda_0)\cap\Lambda$ for all $\omega \in \mathcal{M}$. Clearly, $(D^\alpha_\lambda X_i)$ is a bounded sequence in the Hilbert space $\mathbb{H}_{\varepsilon,\lambda_0,0} = \mathcal{L}^2(S_{\varepsilon,\lambda_0})$ as required by the theorem. Therefore, by passing to a subsequence, we can assume that there exists a weak limit X^α in $\mathcal{L}^2(S_{\varepsilon,\lambda_0})$ such that

$$\lim_{i\to\infty} \int_{B_\varepsilon(\lambda_0)\cap\Lambda} \int_{\mathcal{M}} D^\alpha_\lambda X_i(\lambda,\cdot)\varphi(\lambda,\cdot)\,\mathrm{d}\rho\,\mathrm{d}\lambda = \int_{B_\varepsilon(\lambda_0)\cap\Lambda} \int_{\mathcal{M}} X^\alpha(\lambda,\cdot)\varphi(\lambda,\cdot)\,\mathrm{d}\rho\,\mathrm{d}\lambda.$$

On the other hand

$$\lim_{i\to\infty} \int_{B_\varepsilon(\lambda_0)\cap\Lambda} \int_{\mathcal{M}} X_i(\lambda,\cdot) D^\alpha_\lambda\varphi(\lambda,\cdot)\,\mathrm{d}\rho\,\mathrm{d}\lambda = \int_{B_\varepsilon(\lambda_0)\cap\Lambda} \int_{\mathcal{M}} \tilde{X}(\lambda,\cdot) D^\alpha_\lambda\varphi(\lambda,\cdot)\,\mathrm{d}\rho\,\mathrm{d}\lambda \tag{A.4}$$

by the $\mathcal{L}^2$ convergence of the X_i. This shows weak differentiability of $\tilde{X}$ w.r.t. λ on the set $S_{\varepsilon,\lambda_0}$ and also $X^\alpha = D^\alpha_\lambda \tilde{X}$: Choose any smooth test function $\varphi \in C^\infty_c(B_\varepsilon(\lambda_0)\cap\Lambda)$ and any $A \in \mathcal{A}$, such that $\varphi \mathbf{1}_A \in \mathbb{H}_{\varepsilon,\lambda_0,\delta}$ can be plugged into (A.4). Then use Fubini's theorem and the fact that A is arbitrary.

In particular, we have shown $\tilde{X} \in \mathbb{H}_{\varepsilon,\lambda_0,\delta}$. Since $\varepsilon > 0$, $\lambda_0 \in \Lambda$ can be chosen arbitrarily $\tilde{X}$ is weakly differentiable on the whole of Λ. ✓

Moreover, we can show

$$g(\lambda) := \sum_{1\le|\alpha|\le\delta} \int_{\mathcal{M}} |D^\alpha_\lambda \tilde{X}(\lambda,\cdot)|^2 \,\mathrm{d}\rho \le C,$$

for almost all $\lambda \in B_\varepsilon(\lambda_0)\cap\Lambda$:

Let $B \subseteq B_\varepsilon(\lambda_0)\cap\Lambda$ be measurable. Using weak convergence we get

$$\int_B g(\lambda)\,\mathrm{d}\lambda = \sum_{1\le|\alpha|\le\delta} \int_B \int_{\mathcal{M}} |D^\alpha_\lambda \tilde{X}(\lambda,\cdot)|^2 \,\mathrm{d}\rho\,\mathrm{d}\lambda = \lim_{i\to\infty} \sum_{1\le|\alpha|\le\delta} \int_B \int_{\mathcal{M}} D^\alpha_\lambda X_i(\lambda,\cdot) D^\alpha_\lambda \tilde{X}(\lambda,\cdot)\,\mathrm{d}\rho\,\mathrm{d}\lambda.$$

Using Cauchy-Schwarz' inequality and $\sum_{1\le|\alpha|\le\delta} \int_{\mathcal{M}} |D^\alpha_\lambda X_i(\lambda,\cdot)|^2\,\mathrm{d}\rho \le C$, we obtain

$$\int_B g(\lambda)\,\mathrm{d}\lambda \le \sqrt{|B|\cdot C} \cdot \left(\int_B g(\lambda)\,\mathrm{d}\lambda \right)^{\frac{1}{2}}.$$

In other words $\frac{1}{|B|}\int_B g(\lambda)\,\mathrm{d}\lambda \le C$ for all measurable $B \subseteq B_\varepsilon(\lambda_0)$. This implies $g \le C$ a.e. by Lebesgue's differentiation theorem. ✓

Furthermore, there exists a subsequence $(X_{i_k})_{k\in\mathbb{N}}$ of $(X_i)_{i\in\mathbb{N}}$ such that for every $\alpha \in \mathbb{N}^N$ with $1 \le |\alpha| \le \delta$ the sequence $(D^\alpha_\lambda X_{i_k})_{k\in\mathbb{N}}$ converges to $D^\alpha_\lambda \tilde{X}$ weakly in $\mathcal{L}^2((\Lambda\cap B_l(0))\times\mathcal{M})$ for all $l \in \mathbb{N}$:

According to the above we can choose such a sequence for every fixed $l \in \mathbb{N}$. So, first choose a subsequence s.t. we have weak convergence on $(\Lambda\cap B_1(0))\times\mathcal{M}$. Then choose a subsequence of that subsequence s.t. we also have convergence on $(\Lambda\cap B_2(0))\times\mathcal{M}$, etc. We construct countably many sequences, from which we can take the diagonal sequence, which is, up to finitely many members, a subsequence for every one of these countably many sequences. We, thereby, construct a sequence which converges weakly on every $(\Lambda\cap B_l(0))\times\mathcal{M}$. Also, note that the weak limit of sequences in Hilbert spaces is unique. □

A.3 Chain rules for weak derivatives

Lemma A.3.1. *Let $g\colon \mathbb{R}^N \to \mathbb{R}^m$, where $N, m \in \mathbb{N}$, be Lipschitz continuous. Furthermore, let $X\colon \mathbb{R}^n \to \mathbb{R}^N$, $n \in \mathbb{N}$, be weakly differentiable. Then*

- *$g(X)$ is also weakly differentiable,*
- *for almost every $\lambda \in \mathbb{R}^n$ the restriction $g|_{T_\lambda^X}$ of g to the affine space*

$$T_\lambda^X := \left\{ x \in \mathbb{R}^N \,\middle|\, x = X(\lambda) + \frac{\mathrm{d}}{\mathrm{d}\lambda} X(\lambda) v, \text{ for some } v \in \mathbb{R}^n \right\}$$

  is differentiable at $X(\lambda)$ and

- *for almost all $\lambda \in \mathbb{R}^n$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} g(X)(\lambda) = \frac{\mathrm{d}}{\mathrm{d}x} g|_{T_\lambda^X}(X(\lambda)) \frac{\mathrm{d}}{\mathrm{d}\lambda} X(\lambda).$$

This implies in particular:

- *If $n = N$ and the matrix $\frac{\mathrm{d}}{\mathrm{d}\lambda} X(\lambda)$ is invertible for a.a. λ, then $T_\lambda^X = \mathbb{R}^N$ for a.a. λ and*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} g(X) = \left(\frac{\mathrm{d}}{\mathrm{d}x} g \right) (X) \frac{\mathrm{d}}{\mathrm{d}\lambda} X$$

  a.e., where $\frac{\mathrm{d}}{\mathrm{d}x} g$ is a weak derivative of g.

- *If g is differentiable everywhere, then $\frac{\mathrm{d}}{\mathrm{d}\lambda} g(X) = \left(\frac{\mathrm{d}}{\mathrm{d}x} g \right) (X) \frac{\mathrm{d}}{\mathrm{d}\lambda} X$ a.e.*
- *If g is only locally Lipschitz continuous rather than Lipschitz continuous, but differentiable everywhere, while X is bounded, then still $\frac{\mathrm{d}}{\mathrm{d}\lambda} g(X) = \left(\frac{\mathrm{d}}{\mathrm{d}x} g \right) (X) \frac{\mathrm{d}}{\mathrm{d}\lambda} X$ a.e.*

Proof. For the main statement consult Corollary 3.2 in [AD90]. Concerning the three implications:

- Clearly, if $\frac{\mathrm{d}}{\mathrm{d}\lambda} X(\lambda)$ is invertible for some $\lambda \in \mathbb{R}^n$, then T_λ^X must be the whole $\mathbb{R}^N$ for this λ. So for almost all λ the expression $\frac{\mathrm{d}}{\mathrm{d}x} g|_{T_\lambda^X}(X(\lambda))$ coincides with the classical derivative of g at the point $X(\lambda)$.
  Furthermore, if we choose the identity on $\mathbb{R}^n$ for X, the main statement of the lemma implies that
  - g is differentiable almost everywhere,
  - g is weakly differentiable and
  - any weak derivative of g coincides with the classical derivative up to a null set.

  So, if we define a function on $\mathbb{R}^n$ by setting it to the classical derivative of g at all points for which the classical derivative exists and to 0 for all those points in which it does not, we obtain a weak derivative.

- If g is differentiable everywhere, then $\frac{\mathrm{d}}{\mathrm{d}x} g|_{T_\lambda^X}(X(\lambda))$ is just the classical derivative of g at $X(\lambda)$.
- If X is bounded, we can assume without loss of generality that g is Lipschitz continuous by restricting its domain or using a removable inner cutoff.

□

For the following result let $\mathcal{M}$ be any measurable space and adopt the following notation: For a vector $x \in \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_k}$ let

- $x^{1,i} \in \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_i}$ be the first i components of the vector x if $i \in \{1, \ldots, k\}$ and the empty vector if $i = 0$,
- $x^{i+1,k} \in \mathbb{R}^{N_{i+1}} \times \cdots \times \mathbb{R}^{N_k}$ be the last $k - i$ components of the vector x if $i \in \{0, 1, \ldots, k-1\}$ and the empty vector if $i = k$.

Lemma A.3.2. *Let $g : \mathcal{M} \times \mathbb{R}^N \to \mathbb{R}^m$ be measurable and Lipschitz continuous in the second component, which is further divided via $\mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_k}$ into $k \in \mathbb{N}$ different components. Let L_{g,x_i} be the Lipschitz constant w.r.t. the i-th component for $i = 1, \ldots, k$.*
Furthermore, let $X_i : \mathcal{M} \times \mathbb{R}^n \to \mathbb{R}^{N_i}$, $i = 1, \ldots, k$ be measurable and weakly differentiable w.r.t. $\lambda \in \mathbb{R}^n$. So, $X := (X_1, \ldots, X_k)^\top$ is $\mathbb{R}^N$-valued. Then, the measurable mapping $g(X) : \mathcal{M} \times \mathbb{R}^n \to \mathbb{R}^m$ is also weakly differentiable w.r.t. $\lambda \in \mathbb{R}^n$ and, furthermore, there exist measurable mappings $\Delta^X_{x_i} g : \mathcal{M} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{m \times N_i}$ s.t.

- $\sup_{w \in S^{N_i - 1}} |\Delta^X_{x_i} g(\cdot, \cdot, \cdot) w| \leq L_{g,x_i}$ *everywhere for every $i = 1, \ldots, k$,*
- *for all $v \in \mathbb{R}^n$*
$$\left(\frac{\mathrm{d}}{\mathrm{d}\lambda} g(X)(\omega, \lambda) \right) v = \sum_{i=1}^{k} \left(\Delta^X_{x_i} g(\omega, \lambda, v) \right) \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} X_i(\omega, \lambda) \right) v$$
holds for almost all $\lambda \in \mathbb{R}^n$, $\omega \in \mathcal{M}$.
- *More precisely, $\Delta^X_{x_i} g(\cdot, \lambda, v)$ can be chosen as a measurable cluster point of the bounded sequence*
$$\mathbb{N} \ni l \mapsto \frac{\left(g\left(X^{1,i}(\lambda + \frac{1}{l} v), X^{i+1,k}(\lambda)\right) - g\left(X^{1,i-1}(\lambda + \frac{1}{l} v), X^{i,k}(\lambda)\right)\right) \left(X_i(\lambda + \frac{1}{l} v) - X_i(\lambda)\right)^\top}{|X_i(\lambda + \frac{1}{l} v) - X_i(\lambda)|^2}.$$

In particular:

- *If $N_i = 1$ for some i and, in addition, g is increasing in x_i, then $\Delta^X_{x_i} g \geq 0$ everywhere.*
- *If $n = 1$, we have $\frac{\mathrm{d}}{\mathrm{d}\lambda} g(X) = \sum_{i=1}^{k} \left(\Delta^X_{x_i} g(\cdot, \cdot, 1) \right) \frac{\mathrm{d}}{\mathrm{d}\lambda} X_i$ a.e.*

Proof of Lemma A.3.2. For the weak differentiability of $g(X)$ consult [AD90], Corollary 3.2 (applied ω-wise).
Remember our notation: For a vector $(x_1, \ldots, x_k) \in \mathbb{R}^{\sum_{i=1}^{k} N_i}$, $x^{i,j}$ refers to the vector $(x_i, \ldots, x_j)$, where $i, j \in \{1, \ldots, k\}$ and $i \leq j$.
Let $v \in S^{n-1}$ be fixed. Let $\lambda \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $\omega \in \Omega$. Then

$$\begin{aligned}
&g(\omega, X(\omega, \lambda + tv)) - g(\omega, X(\omega, \lambda)) = \\
&\quad = \sum_{i=1}^{k} \left(g\left(\omega, X^{1,i}(\omega, \lambda + tv), X^{i+1,k}(\omega, \lambda)\right) - g\left(\omega, X^{1,i-1}(\omega, \lambda + tv), X^{i,k}(\omega, \lambda)\right) \right) = \\
&= \sum_{i=1}^{k} \frac{\left(g\left(X^{1,i}(\lambda + tv), X^{i+1,k}(\lambda)\right) - g\left(X^{1,i-1}(\lambda + tv), X^{i,k}(\lambda)\right)\right) \left(X_i(\lambda + tv) - X_i(\lambda)\right)^\top}{|X_i(\lambda + tv) - X_i(\lambda)|^2} (\omega) \cdot \\
&\qquad \cdot \left(X_i(\omega, \lambda + tv) - X_i(\omega, \lambda) \right), \quad \text{(A.5)}
\end{aligned}$$

where we use the convention $\frac{0}{0} := 0$. Now, define

$$\Delta_{x_i}^{X,l} g(\cdot, \lambda, v) :=$$
$$= \frac{\left(g\left(X^{1,i}(\lambda + \frac{1}{l}v), X^{i+1,k}(\lambda)\right) - g\left(X^{1,i-1}(\lambda + \frac{1}{l}v), X^{i,k}(\lambda)\right)\right)\left(X_i(\lambda + \frac{1}{l}v) - X_i(\lambda)\right)^\top}{|X_i(\lambda + \frac{1}{l}v) - X_i(\lambda)|^2}.$$

Note that $\sup_{w \in S^{N_i - 1}} \left|\Delta_{x_i}^{X,l} g\right|_w$ can be assumed to be bounded by L_{g,x_i} (everywhere), due to Lipschitz continuity of g in the i-th component (we can assume without loss of generality that g is truly Lipschitz continuous for all ω). Furthermore, $\Delta_{x_i}^{X,l} g$ is clearly a measurable mapping on $\mathcal{M} \times \mathbb{R}^n \times \mathbb{R}^n$. We have, therefore, a sequence $\left(\left(\Delta_{x_i}^{X,l} g\right)_{i=1,\dots,k}\right)_{l \in \mathbb{N}}$ of uniformly bounded measurable $\mathbb{R}^{m \times N}$-valued mappings on $\mathcal{M} \times \mathbb{R}^n \times \mathbb{R}^n$. Therefore, there must exist a cluster point $\left(\Delta_{x_i}^{X} g\right)_{i=1,\dots,k}$ of this sequence, which can be selected in such a way that it describes a measurable mapping (e.g. follow the standard proof of the Bolzano-Weierstrass theorem and check that measurability is preserved in every step of the construction). The property $\sup_{w \in S^{N_i - 1}} \left|\Delta_{x_i}^{X} g\right|_w \leq L_{g,x_i}$ will obviously be inherited.
Now, note that for almost all λ, ω

$$\lim_{t \to 0} \frac{g(\omega, X(\omega, \lambda + tv)) - g(\omega, X(\omega, \lambda))}{t} = \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} g(X)(\omega, \lambda)\right) v,$$

$$\lim_{t \to 0} \frac{X_i(\omega, \lambda + tv) - X_i(\omega, \lambda)}{t} = \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} X_i(\omega, \lambda)\right) v,$$

which is a consequence of Lemma A.2.1 and the fundamental Theorem of Lebesgue integral calculus. Recall that v is fixed here. Now, for almost all λ, ω we can choose a sequence $(t_q)_{q \in \mathbb{N}}$, which is a subsequence of $\left(\frac{1}{l}\right)_{l \in \mathbb{N}}$ with

$$\Delta_{x_i}^{X} g(\omega, \lambda, v) =$$
$$= \lim_{q \to \infty} \frac{\left(g\left(X^{1,i}(\lambda + t_q v), X^{i+1,k}(\lambda)\right) - g\left(X^{1,i-1}(\lambda + t_q v), X^{i,k}(\lambda)\right)\right)\left(X_i(\lambda + t_q v) - X_i(\lambda)\right)^\top}{|X_i(\lambda + t_q v) - X_i(\lambda)|^2}(\omega)$$

and, at the same time,

$$\lim_{q \to \infty} \frac{g(\omega, X(\omega, \lambda + t_q v)) - g(\omega, X(\omega, \lambda))}{t_q} = \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} g(X)(\omega, \lambda)\right) v,$$

$$\lim_{q \to \infty} \frac{X_i(\omega, \lambda + t_q v) - X_i(\omega, \lambda)}{t_q} = \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} X_i(\omega, \lambda)\right) v.$$

So, by plugging in t_q for t in (A.5), dividing the resulting expression by t_q and letting $q \to \infty$ we obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}\lambda} g(X)(\omega, \lambda)\right) v = \sum_{i=1}^{k} \left(\Delta_{x_i}^{X} g(\omega, \lambda, v)\right) \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} X_i(\omega, \lambda)\right) v$$

for almost all $(\omega, \lambda) \in \Omega \times \mathbb{R}^n$. □

A.4 Miscellaneous

The following statement should be known:

Lemma A.4.1. *Let $\mathbb{Q} \sim \mathbb{P}$ be a probability measure. Define $R := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ as the Radon-Nikodym derivative. Then the martingale $R_t := \mathbb{E}[R|\mathcal{F}_t]$ can be written as*

$$R_t = \exp\left(\int_0^t \zeta_s \,\mathrm{d}W_s - \frac{1}{2}\int_0^t |\zeta_s|^2 \,\mathrm{d}s\right),$$

with some progressively measurable process ζ s.t. $\int_0^T |\zeta_s|^2 \,\mathrm{d}s < \infty$ a.s.

Proof. Using Itô's martingale representation theorem we can w.l.o.g assume that $R.$ is a continuous martingale and write

$$R_t = 1 + \int_0^t \eta_s \,\mathrm{d}W_s$$

with some progressive η s.t. $\int_0^T |\eta_s|^2 \,\mathrm{d}s < \infty$ a.s.
Since $R_T > 0$ a.s. we also have $R_t = \mathbb{E}[R_T|\mathcal{F}_t] > 0$ a.s. for all $t \in [0, T]$. Using continuity of $R.$ this means $\inf_t R_t > 0$ a.s. This implies $\sup_t \frac{1}{R_t} < \infty$ a.s. and, hence, $\int_0^T |\zeta_s|^2 \,\mathrm{d}s < \infty$ a.s. for $\zeta_t := \frac{\eta_t}{R_t}$. We also have

$$R_t = 1 + \int_0^t R_s \zeta_s \,\mathrm{d}W_s.$$

But this already implies

$$R_t = \exp\left(\int_0^t \zeta_s \,\mathrm{d}W_s - \frac{1}{2}\int_0^t |\zeta_s|^2 \,\mathrm{d}s\right).$$

(Apply Itô formula to $\ln(R_t)$.) □

The following approximation result will be needed for Lemma A.4.3:

Lemma A.4.2. *Let $\mathbb{Q}$ be a probability measure on an arbitrary measurable space $(\Omega, \mathcal{F})$. Let Y be non-negative and $\mathcal{F}$-measurable. Let $p, q > 1$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\mathbb{E}_\mathbb{Q}[Y^p]^{\frac{1}{p}} = \sup_{X \in \mathcal{X}} \frac{\mathbb{E}_\mathbb{Q}[Y \cdot X]}{\mathbb{E}_\mathbb{Q}[X^q]^{\frac{1}{q}}},$$

where $\mathcal{X}$ is the set of all measurable $X : \Omega \to \mathbb{R}$ such that there exist constants $C_1, C_2 > 0$ with $0 < C_1 \le X(\omega) \le C_2$ for all $\omega \in \Omega$.

Proof. Firstly, using Hölder-inequality:

$$\frac{\mathbb{E}_\mathbb{Q}[Y \cdot X]}{\mathbb{E}_\mathbb{Q}[X^q]^{\frac{1}{q}}} \le \frac{(\mathbb{E}_\mathbb{Q}[Y^p])^{\frac{1}{p}} (\mathbb{E}_\mathbb{Q}[X^q])^{\frac{1}{q}}}{\mathbb{E}_\mathbb{Q}[X^q]^{\frac{1}{q}}},$$

which implies

$$\sup_{X \in \mathcal{X}} \frac{\mathbb{E}_\mathbb{Q}[Y \cdot X]}{\mathbb{E}_\mathbb{Q}[X^q]^{\frac{1}{q}}} \le \mathbb{E}_\mathbb{Q}[Y^p]^{\frac{1}{p}}.$$

Secondly, defining $X_{nm} := \left(\frac{1}{m} + Y \wedge n\right)^{p-1}$, which is bounded and positive, and assuming that $0 < \mathbb{E}_\mathbb{Q}[Y] < \infty$ for a moment, we can apply monotone convergence:

$$\sup_{0 < X \in L^\infty(\mathcal{F})} \frac{\mathbb{E}_\mathbb{Q}[Y \cdot X]}{(\mathbb{E}_\mathbb{Q}[X^q])^{\frac{1}{q}}} \ge \sup_{n,m} \frac{\mathbb{E}_\mathbb{Q}[Y \cdot X_{nm}]}{(\mathbb{E}_\mathbb{Q}[X_{nm}^q])^{\frac{1}{q}}} \ge \lim_{n\to\infty} \lim_{m\to\infty} \frac{\mathbb{E}_\mathbb{Q}[(Y \wedge n) \cdot X_{nm}]}{(\mathbb{E}_\mathbb{Q}[X_{nm}^q])^{\frac{1}{q}}} =$$

$$= \lim_{n\to\infty} \frac{\mathbb{E}_\mathbb{Q}[(Y \wedge n) \cdot (Y \wedge n)^{p-1}]}{(\mathbb{E}_\mathbb{Q}[(Y \wedge n)^{(p-1)q}])^{\frac{1}{q}}} = \lim_{n\to\infty} (\mathbb{E}_\mathbb{Q}[(Y \wedge n)^p])^{\frac{1}{p}} = (\mathbb{E}_\mathbb{Q}[Y^p])^{\frac{1}{p}},$$

where we used $\mathbb{E}_\mathbb{Q}[(Y \wedge n)^p] > 0$ for n large enough. We also used $(p-1)q = p$.
Now, if $\mathbb{E}_\mathbb{Q}[Y] = 0$, the proof is trivial, since this would imply $Y = 0$ a.s.
If $\mathbb{E}_\mathbb{Q}[Y] = \infty$, the proof becomes trivial as well, since then $\mathbb{E}_\mathbb{Q}[Y^p] = \infty$ and we can set $X = 1$. □

Lemma A.4.3. *Let $(\mathbb{Q}_n)_{n\in\mathbb{N}}$ be a sequence of probability measures such that $\mathbb{Q}_n \sim \mathbb{P}$ and for some $p > 1$:*

$$\sup_n \mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{P}}\right)^p\right] < \infty. \tag{A.6}$$

Assume furthermore

$$\frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{P}} = \exp\left(\int_0^T \zeta_s^n \mathrm{d}W_s - \frac{1}{2}\int_0^T |\zeta_s^n|^2 \mathrm{d}s\right)$$

such that

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^T |\zeta_s^n|\,\mathrm{d}s\right)^q\right] = 0,$$

where $\frac{1}{q} + \frac{1}{p} = 1$. Then

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}[X] = \mathbb{E}[X] \qquad \forall X \in L^q(\mathbb{P}). \tag{A.7}$$

Proof. Set $R^n := \frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{P}}$. Then R^n is a bounded sequence in $L^p(\mathbb{P}) = L^p(\mathcal{F}_T, \mathbb{P})$ according to (A.6).

Firstly, we claim that in order to prove the lemma it is actually sufficient to show that all subsequences of $(\mathbb{Q}_n)$ have $\mathbb{P}$ as a cluster point (w.r.t. convergence used in (A.7)).

Proof: Assume the latter has been shown. Now, take any $X \in L^q(\mathbb{P})$. Then $\mathbb{E}_{\mathbb{Q}_n}[X] = \mathbb{E}[R^n X]$ is a bounded sequence (using the Hölder inequality). For the limes superior of the sequence there exists a subsequence $\mathbb{Q}_{n_k}$ s.t. $\mathbb{E}_{\mathbb{Q}_{n_k}}[X]$ converges to the limes superior. But by extracting a subsequence of $(\mathbb{Q}_{n_k})$ the corresponding subsequence of $\mathbb{E}_{\mathbb{Q}_{n_k}}[X]$ would converge to $\mathbb{E}_{\mathbb{P}}[X]$. Hence, $\mathbb{E}_{\mathbb{P}}[X]$ is equal to the limes superior of $\mathbb{E}_{\mathbb{Q}_n}[X]$. Similarly the limes inferior would also be equal to $\mathbb{E}_{\mathbb{P}}[X]$. ✓

Since (R^n) (or any subsequence of (R^n)) is a bounded sequence in $L^p(\mathbb{P})$, there exists an $R \in L^p(\mathbb{P})$ s.t. $\mathbb{E}[R^p] \le \sup_n \mathbb{E}[(R^n)^p]$ together with a subsequence of (R^n) (or of any subsequence of (R^n)), which we again denote by (R^n) (by a slight abuse of notation), which converges to R, i.e. $\lim_{n\to\infty} \mathbb{E}[R^n X] = \mathbb{E}[RX]$ for all $X \in L^q(\mathbb{P})$ with $\frac{1}{q} + \frac{1}{p} = 1$. This works because $L^p(\mathbb{P})$ is a reflexive Banach space. So, in other words we assume without loss of generality that R^n converges to R in the above sense.
We want to show that $R = 1$ a.s.
From this convergence we get immediately that $\mathbb{E}[RX] \ge 0$ for all non-negative and bounded $X \in L^\infty(\mathcal{F}_T)$, which implies that R is a.s. non-negative: setting $X = \mathbf{1}_{\{R<0\}}$ we have $R\mathbf{1}_{\{R<0\}} = 0$ a.s.. Furthermore, setting $X = 1$ we get $\mathbb{E}[R] = 1$. This means that $\mathbb{Q} := R \cdot \mathbb{P}$ is a probability measure.

Claim1: $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and $\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\right)^p\right] < \infty$.
Proof: We have using Lemma A.4.2

$$\left(\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{1}{R}\right)^p\right]\right)^{\frac{1}{p}} = \sup_{X\in\mathcal{X}} \frac{\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{R}X\right]}{\left(\mathbb{E}_{\mathbb{Q}}\left[|X|^q\right]\right)^{1/q}} = \sup_{X\in\mathcal{X}} \frac{\mathbb{E}\left[X\right]}{\left(\mathbb{E}\left[R|X|^q\right]\right)^{1/q}} =$$

$$= \sup_{X\in\mathcal{X}} \lim_{n\to\infty} \frac{\mathbb{E}\left[X\right]}{\left(\mathbb{E}\left[R^n|X|^q\right]\right)^{1/q}} \le \sup_{X\in\mathcal{X}} \sup_n \frac{\mathbb{E}\left[X\right]}{\left(\mathbb{E}\left[R^n|X|^q\right]\right)^{1/q}} =$$

$$= \sup_n \sup_{X\in\mathcal{X}} \frac{\mathbb{E}_{\mathbb{Q}_n}\left[\frac{1}{R^n}X\right]}{\left(\mathbb{E}_{\mathbb{Q}_n}\left[|X|^q\right]\right)^{1/q}} = \sup_n \left(\mathbb{E}_{\mathbb{Q}_n}\left[\left(\frac{1}{R^n}\right)^p\right]\right)^{\frac{1}{p}} < \infty$$

This means $\mathbb{E}\left[\frac{1}{R^{p-1}}\right] = \mathbb{E}\left[R\left(\frac{1}{R}\right)^p\right] < \infty$ and, therefore, $R > 0$ a.s. and we have $\mathbb{Q} \sim \mathbb{P}$. ✓

Now, set $R_t^n := \mathbb{E}[R^n|\mathcal{F}_t]$ and write

$$R_t^n = \exp\left(\int_0^t \zeta_s^n \mathrm{d}W_s - \frac{1}{2}\int_0^t |\zeta_s^n|^2 \mathrm{d}s\right)$$

for some progressively measurable ζ^n. And similarly

$$R_t = \exp\left(\int_0^t \zeta_s \mathrm{d}W_s - \frac{1}{2}\int_0^t |\zeta_s|^2 \mathrm{d}s\right)$$

with $R_t := \mathbb{E}[R|\mathcal{F}_t]$ (Lemma A.4.1). Also, note $R^n_T = R^n$ and $R_T = R$.
We now claim that ζ^n converges to ζ in some weak sense:

Claim2:

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}\left[\int_0^T \zeta^n_s \cdot \lambda_s \,\mathrm{d}s\right] = \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \zeta_s \cdot \lambda_s \,\mathrm{d}s\right]$$

for all progressively measurable $\mathbb{R}^d$-valued λ s.t. $\mathbb{E}\left[\left(\int_0^T |\lambda_s|^2 \,\mathrm{d}s\right)^{\frac{q}{2}}\right] < \infty$.

Proof: Using the Itô formula

$$R^n_T = 1 + \int_0^T R^n_s \zeta^n_s \,\mathrm{d}W_s \tag{A.8}$$

and

$$R_T = 1 + \int_0^T R_s \zeta_s \,\mathrm{d}W_s.$$

Now, define $X := \int_0^T \lambda_s \,\mathrm{d}W_s$ and $X_t := \int_0^t \lambda_s \,\mathrm{d}W_s$. Then $\mathbb{E}[R^n_T X]$ converges to $\mathbb{E}[R_T X]$, since $X \in L^q(\mathbb{P})$, which follows using BDG-inequalities. On the other hand we can calculate the cross-variation of the $\mathbb{P}$ - martingales $R^n_\cdot$ and $X_\cdot$:

$$\langle R^n_\cdot, X_\cdot \rangle_t = \int_0^t R^n_s \zeta^n_s \lambda_s \,\mathrm{d}s.$$

Therefore,

$$\mathbb{E}[R^n_{\tau_k} X_{\tau_k}] = \mathbb{E}\left[\int_0^{\tau_k} R^n_s \zeta^n_s \lambda_s \,\mathrm{d}s\right]$$

for some localizing sequence of stopping times (τ_k).
Using (A.8), $\sup_n \mathbb{E}[(R^n_T)^p] < \infty$ and BDG-inequalities we have

$$\sup_n \mathbb{E}\left[\left(\int_0^T |R^n_s \zeta^n_s|^2 \,\mathrm{d}s\right)^{\frac{p}{2}}\right] < \infty, \tag{A.9}$$

which implies $\mathbb{E}\left[\int_0^T |R^n_s \zeta^n_s \lambda_s| \,\mathrm{d}s\right] < \infty$ (using Cauchy-Schwarz and Hölder inequalities). Using dominated convergence (for $k \to \infty$) this implies

$$\mathbb{E}[R^n_T X] = \mathbb{E}\left[\int_0^T R^n_s \zeta^n_s \lambda_s \,\mathrm{d}s\right],$$

where we in particular use, that as a corollary of Doob's inequality $\sup_{t\in[0,T]} |R^n_t| \in L^p(\mathbb{P})$ and $\sup_{t\in[0,T]} |X_t| \in L^q(\mathbb{P})$, such that their product is integrable.
Similarly

$$\mathbb{E}[R_T X] = \mathbb{E}\left[\int_0^T R_s \zeta_s \lambda_s \,\mathrm{d}s\right].$$

Furthermore,

$$\mathbb{E}\left[R^n_T X\right] = \mathbb{E}\left[\int_0^T R^n_s \zeta^n_s \lambda_s \,\mathrm{d}s\right] = \int_0^T \mathbb{E}[R^n_s \zeta^n_s \lambda_s] \,\mathrm{d}s = \int_0^T \mathbb{E}[[R^n_T|\mathcal{F}_s] \zeta^n_s \lambda_s] \,\mathrm{d}s =$$

$$= \int_0^T \mathbb{E}[\mathbb{E}[R_T^n \zeta_s^n \lambda_s | \mathcal{F}_s]] \,\mathrm{d}s = \mathbb{E}\left[\int_0^T R_T^n \zeta_s^n \lambda_s \,\mathrm{d}s\right] = \mathbb{E}_{\mathbb{Q}_n}\left[\int_0^T \zeta_s^n \lambda_s \,\mathrm{d}s\right]$$

And similarly $\mathbb{E}[R_T X] = \mathbb{E}\left[\int_0^T R_s \zeta_s \lambda_s \,\mathrm{d}s\right] = \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \zeta_s \lambda_s \,\mathrm{d}s\right]$.
Now, the assertion follows using the convergence of $\mathbb{E}[R_T^n X]$ to $\mathbb{E}[R_T X]$. ✓

In order to finish the proof we need to show $\mathbb{Q} = \mathbb{P}$, or $R = 1$ a.s.. This is equivalent to $\zeta = 0$ a.e.. For this $\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \zeta_s \cdot \lambda_s \,\mathrm{d}s\right] = 0$ for all bounded λ would be sufficient. Using Claim2 we only need to show $\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}\left[\int_0^T \zeta_s^n \cdot \lambda_s \,\mathrm{d}s\right] = 0$:

$$\left|\mathbb{E}_{\mathbb{Q}_n}\left[\int_0^T \zeta_s^n \cdot \lambda_s \,\mathrm{d}s\right]\right| = \left|\mathbb{E}\left[R^n \int_0^T \zeta_s^n \cdot \lambda_s \,\mathrm{d}s\right]\right| \leq$$

$$\leq (\mathbb{E}[(R^n)^p])^{\frac{1}{p}} \left(\mathbb{E}\left[\left(\int_0^T |\zeta_s^n||\lambda_s| \,\mathrm{d}s\right)^q\right]\right)^{\frac{1}{q}} \leq \sup_{n\in\mathbb{N}} (\mathbb{E}[(R^n)^p])^{\frac{1}{p}} \|\lambda\|_\infty \left(\mathbb{E}\left[\left(\int_0^T |\zeta_s^n| \,\mathrm{d}s\right)^q\right]\right)^{\frac{1}{q}},$$

which converges to 0 for $n \to \infty$. $\square$

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Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 07.07.14

Alexander Fromm